Lecture 2 : Linear programming and convex optimization

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We talked about optimization problems and why they are important. We also looked at one of the class of problems called least square problems. Next we look at another class,

1 Linear Programming

1.1 Definition

Linear programming is one of the well studied classes of optimization problem. We already discussed that a linear program is one which has linear objective and constraint functions. This implies that a standard linear program looks like

$$\begin{array}{rl} \min & \sum_i c_i x_i = c^T x \\ \text{subject to} & a_i^T x_i \leq b_i \quad \forall i \in \{1, \cdots, m\} \end{array}$$

Here the vectors $c, a_1, \cdots, a_m \in \mathbb{R}^n$ and scalars $b_i \in \mathbb{R}$ are the problem parameters.

1.2 Examples

– Max flow: Given a graph, start(s) and end node (t), capacities on every edge; find out the maximum flow possible through edges.

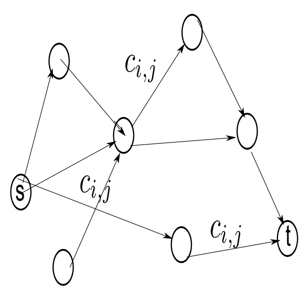

Fig. 1. Max flow problem: there will be capacities for every edge in the problem statement

The linear program looks like:

$$\begin{array}{ll} \max & \sum_{\{s,u\}} f(s,u) \\ \text{s.t.} & \sum_{\{u,v\}} f(u,v) = \sum_{\{v,u\}} f(v,u)\ \forall v \neq s,t, \sim 0 \leq f(u,v) \leq c(u,v) \end{array}$$

Note: There is another one which can be made using the flow through paths.

* Thanks to books from Boyd and Vandenberghe, Dantzig and Thapa, Papadimitriou and Steiglitz

- Another example: This time, I will give the linear program and you will tell me what real world situation models it :).

$$\begin{aligned} \max \quad & 2x_1 + 4x_2 \\ \text{s.t. } & x_1 + x_2 \leq 10,\ x_1 \leq 4 \end{aligned}$$

1.3 Solving linear programs

You might have already had a course on linear optimization. So you know that there are many known algorithms for solving linear programs; like simplex, ellipsoid and interior point method. Ellipsoid method was the first known polynomial time algorithm and now interior point method (also polynomial) is used in many places to solve real world linear programs. We are not going to see how this algorithms/methods work. The lesson is that if we can formulate any problem as a linear program, it is "solved".

Because of the abundance of algorithms to solve linear programs, researchers were really excited about this paradigm. There were many attempts to solve even NP hard problems (like traveling salesman problem) using linear programming. Notice that this will prove one of the most fundamental questions of complexity theory, P=NP. This is because we know that linear programs can be solved in polynomial time.

Recently there was a big result by Wolf et. al. where they showed that most of these techniques are bound to fail. They showed that the traveling salesman polytope or its extension will require exponential number of constraints. This will be one of the choices for project.

1.4 Formulating a problem as linear program

From previous discussion it is quite clear that we want to see what problems can be formulated as a linear program. Unfortunately, it is not as clear as in the case of least square problems. I told you that it is easy to check if the objective function is of least square form (not proven !!, we learn the basics later). How about linear programs? Say the standard form of linear programs is (This choice is arbitrary and is chosen because it is convenient for following examples).

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t. } & a_i^T x = b_i \ \forall i \in [m] \\ & x_i \geq 0 \quad \forall i \in [n] \end{aligned}$$

For linear programs, we have few standard tricks by which we can convert a problem into a linear program (in the standard form). Lets look at few cases below.

- Converting min to max
- From inequality to equality, slack variables.
- Converting negative variables into positive
- From real variables to positive variables.
- This problem is called Chebyshev approximation:

$$\min_x \max_i |a_i^T x - b_i|$$

  This looks like least squares. But it can actually be formulated using linear programming.

$$\begin{aligned} \max \quad & t \\ \text{s.t. } & a_i^T x - b_i \leq t,\ a_i^T x + b_i \leq t \end{aligned}$$

2 Convex optimization

Convex optimization is a generalization of linear programming where the constraints and objective function are convex. Both the least square problems and linear programming is a special case of convex optimization. Can we prove this?

It is interesting because most of the algorithms for linear programming can be generalized to convex optimization too. More importantly, many more problems can be expressed in this framework than linear programming. In next few weeks, we will cover the basics about convex functions and their geometry. This will help us when we actually see convex optimization in action.