Lecture 19: Properties of theta function

Rajat Mittal

IIT Kanpur*

The Lovasz theta number was introduced last time to find out upper bounds on the size of independent set. For the case of perfect graphs (chromatic number is same as maximum clique size), the Lovasz theta number of complementary graph gives us the maximum independent set size.

Though for general graphs it has been proved that Lovasz theta number is not a tight bound on the size of maximum independent set size. In spite of this the theta function is useful because it is an SDP and using SDP structure we can come up with lot of properties of independent set and chromatic number.

To remind, the Lovasz theta number for a graph G is given by following SDP's.

Primal

$$\begin{aligned} &\max \textstyle\sum_{(i,j)\in E} X_{ij} \\ \text{s.t.}\quad & X_{ij} = 0 \ \ \forall (i,j) \in E \\ &\textstyle\sum_{i\in V} Tr(X) = 1 \\ & X \succeq 0 \end{aligned} \tag{1}$$

Dual

$$\begin{aligned} &\min \lambda \\ \text{s.t.}\quad & \lambda I - A \succeq 0 \\ & A_{i,j} = 1 \text{ if } i = j \text{ or } (i,j) \notin E \end{aligned} \tag{2}$$

1 Shannon capacity of a graph

Shannon introduced *Shannon capacity* of a graph to estimate the capacity of a channel with noise. Suppose there is a channel and a list of symbols (say $a, b, \cdots$) which can be sent across the channel. Since the channel is noisy, some of the pairs can be confused when transferred across the channel.

The effect of this noise can be modeled by a *confusability graph* G. The graph G has vertices for every alphabet and there is an edge between two vertices if they can be confused by the receiver of the channel. Clearly the maximum number of symbols which can be sent is the maximum independent set of this graph.

A better strategy would be to send multiple letters so that the chances of confusion are less, i.e., our new symbol set is $aa, ab, bc, \cdots$. In this case the new confusability graph will be related to the original confusability graph. The relation is given by *strong* product of a graph.

Given two graphs $G_1 = (V_1, E_1), G_2 = (V_2, E_2)$, the strong product $G_1 \times G_2$ is defined as the graph with vertex set $V_1 \times V_2$. The edges are defined by relation,

$$(i_1, i_2), (j_1, j_2) \in E(G_1 \times G_2) \Leftrightarrow (i_1 = j_1 \vee (i_1, j_1) \in E_1) \wedge (i_2 = j_2 \vee (i_2, j_2) \in E_2).$$

Exercise 1. Convince yourself that the confusability graph when we send two symbols is the strong product of confusability graph when we send just one symbol.

Note: The strong product can be thought of as a product where edges are present in the new graph if for *all* co-ordinates there is an edge or both the vertices are equal. There is a *weak* product also where in the newly formed graph the edges are present if for *any* co-ordinate there is an edge in the original graph. The strong product and weak product are useful in many applications and not just Shannon capacity.

With this product the number of symbols we can send are $\alpha(G^2)$. To normalize the cost (sending two symbols instead of one), we compare the square root of $\alpha(G^2)$ with $\alpha(G)$.

Exercise 2. Show that $\alpha(G) \leq \sqrt{\alpha(G^2)}$. In general show that if $k \leq l$ then $\sqrt[k]{\alpha(G^k)} \leq \sqrt[l]{\alpha(G^l)}$.

* Lovasz's paper on Shannon capacity

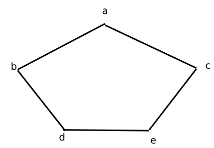

Fig. 1. Example of a confusability graph

This idea can be expanded further and we can consider the channel capacity as the maximum of $\sqrt[k]{\alpha(G^k)}$. The *Shannon capacity* is defined by,

$$\lim_{k\to\infty} \sqrt[k]{\alpha(G^k)}.$$

2 The product property of theta number

From the definition of Shannon capacity, it is clear that the quantity $\theta(G^k)$ is interesting. So, the natural question is how $\theta(G_1 \times G_2)$ is related to $\theta(G_1), \theta(G_2)$.

Theorem 1. *For graphs G_1 and G_2,*

$$\theta(G_1 \times G_2) = \theta(G_1)\theta(G_2).$$

We will not show the proof but comment that one of the way to show this theorem is using the structure of SDP's. In general it can be asked, given the description (C, A_i, b) of SDP, what happens when the to the optimal value if a product between two SDP's are taken. The product of SDP's can be defined in many ways, but one of the most natural one is when we take tensor product of the respective parameters. This product turns out to be useful in many applications.

Suppose the primal SDP's are S_1, S_2 and their duals D_1, D_2. Denote the product primal SDP by $S_1 \otimes S_2$ and the dual of it by $D(S_1 \otimes S_2)$.The outline of the proof is,

1. Show that for the optimal solution X_1^*, X_2* of S_1, S_2 respectively, $X_1^* \otimes X_2^*$ is the solution of $S_1 \otimes S_2$ with the objective value $Opt(S_1)\ Opt(S_2)$. This implies $Opt(S_1 \otimes S_2) \geq Opt(S_1)\ Opt(S_2)$.
2. Show that for the optimal solution y_1^*, y_2* of D_1, D_2 respectively, $y_1^* \otimes y_2^*$ is the solution of $D(S_1 \otimes S_2)$ with the objective value $Opt(D_1)\ Opt(D_2)$. This implies $Opt(D(S_1 \otimes S_2)) \geq Opt(D_1)\ Opt(D_2)$.
3. Using strong duality for all the SDP's $Opt(S_1 \otimes S_2) = Opt(S_1)\ Opt(S_2)$.

Exercise 3. Use the above strategy to show the theorem.

Using this product property of Lovasz theta number (Thm. 1),

Exercise 4. Show that the Shannon capacity of the graph is less than $\theta(G)$.

Lovasz showed the Shannon capacity of the pentagon (C_5) using the above exercise. He showed that $\theta(C_5) = \sqrt{5} = \sqrt{\alpha(C_5^2)}$. Hence,

$$\sqrt{\alpha(C_5^2)} \leq \lim_{k \to \infty} \sqrt[k]{\alpha(C_5^k)} \leq \theta(C_5) = \sqrt{\alpha(C_5^2)}.$$

So the Shannon capacity of pentagon is $\sqrt{5}$.

3 Theta number of complementary graph

There was another formulation of Lovasz theta number in the last lecture.

$$\begin{aligned} & \min_{c,\{v_i\}} \max_i \left(\tfrac{1}{c^T v_i}\right)^2 \\ \text{s.t.} \quad & \|c\| = \|v_i\| = 1 \ \ \forall i \\ & v_i^T v_j = 0 \ \ \forall (i,j) \notin E \end{aligned} \tag{3}$$

Using this characterization we can prove,

Theorem 2. *For a graph G and its complement $\bar{G}$,*

$$\theta(G)\theta(\bar{G}) \geq n.$$

Proof. Suppose $c, \{v_i\}$ and $d, \{w_i\}$ are the optimal vectors for G and $\bar{G}$ respectively. Then the unit vectors $v_i \otimes w_i$ and $v_j \otimes w_j$ are orthogonal to each other iff $i \neq j$. So,

$$1 \geq \sum_i ((c \otimes d)^T (v_i \otimes w_i))^2 = \sum_i (c^T v_i)^2 (d^T w_i)^2.$$

From the fact that $c, \{v_i\}$ is feasible for $\theta(G)$,

$$1 \geq \sum_i \frac{(d^T w_i)^2}{\theta(G)} \Rightarrow \theta(G) \geq \sum_i (d^T w_i)^2.$$

Now using the fact that $d, \{w_i\}$ is feasible for $\theta(\bar{G})$,

$$\theta(G) \geq \sum_i \frac{1}{\theta(\bar{G})} \Rightarrow \theta(G)\theta(\bar{G}) \geq n.$$

□