Lecture 16: Strong duality

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In the last lecture we saw how to formulate a dual program for an optimization problem. From the formulation, it turned out that the dual program gave an upper/lower bound depending on whether the problem was maximization/minimization respectively.

Today, the focus will be on when are these bounds tight. It turns out that these bounds are tight in most of the cases of interest like linear programming and semidefinite programming. The previous statement is almost true except in few contrived cases. When the primal and dual values agree, *strong duality* is said to hold. We will look at the conditions under which strong duality holds.

Lets look at the primal-dual pair for semidefinite programming.

Primal	Dual
$\max C \bullet X$ s.t. $A_i \bullet X = b_i \quad \forall i$ $X \in \mathcal{S}_n$	$\min y^T b$ s.t. $\sum_i y_i A_i - C \in \mathcal{S}_n$

First, it is instructive to see the direct proof of weak duality. Given any feasible solution X for primal and y for dual,

$$C \bullet X \leq (\sum_i y_i A_i) \bullet X = \sum_i y_i b_i = y^T b. \tag{1}$$

Notice that the above equation implies, all feasible solutions of dual give an upper bound on the optimal value of primal. Similarly any feasible solution for primal gives a lower bound on the optimal value of dual.

Suppose the optimal value of primal is p^*, attained at X^*. Similarly the optimal value of dual is d^* and obtained for y^*. Weak duality implies that $p^* \leq d^*$. Assume that this two values are equal, i.e., $p^* = d^*$ (strong duality). Then from Eqn. 1,

$$\begin{aligned} C \bullet X &= (\textstyle\sum_i y_i A_i) \bullet X \\ \Rightarrow (\sum_i y_i A_i - C) \bullet X &= 0 \end{aligned}$$

The condition above is called the *complementary slackness* condition. So, for optimal X^*, y^* with strong duality, the complementary slackness condition holds. Conversely, if X, y are feasible solutions of primal and dual respectively and satisfy the complementary slackness condition then strong duality holds and $p^* = d^*$.

Exercise 1. If for two matrices $M, N \succeq 0$, say $M \bullet N = 0$. Then show that the eigenvectors of M, N corresponding to non-zero eigenvalues are orthogonal to each other.

Exercise 2. What does the complementary slackness condition tell us in case of linear programming?

The discussion above is done for semidefinite program, but with little more effort can be generalized to cone programs.

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[*] Thanks to Boyd and Vandenberghe's book on convex optimization

1 Strong duality: Slater's condition

It turns out that in most of the applications of semidefinite programming to real world, strong duality holds. Hence the optimal value of primal is same as optimal value of dual. Strong duality can be obtained by verifying *Slater condition*. Specifically, if the semidefinite program satisfies *Slater conditions* then it has strong duality.

Theorem 1. *Given a semidefinite program in standard form with parameters C, A_i, b, suppose the feasible set of primal is $\mathcal{P}$ and feasible set of dual is $\mathcal{D}$. Then strong duality holds if either*

- *If $\mathcal{D} \neq \varnothing$ and there exists a strictly feasible $X \in \mathcal{P}$, i.e., $X \succ 0, A_i \bullet X = b_i \ \ \forall i$.*
  or
- *If $\mathcal{P} \neq \varnothing$ and there exists a strictly feasible $y \in \mathcal{D}$, i.e., $\sum_i y_i A_i - C \succ 0$.*

In other words, if the primal is feasible and dual is strictly feasible or vice versa then strong duality holds. In the usual cases, the feasible region would be expected to be non-empty and even have non-empty interior. That would imply strong duality from Slater's condition.

Proof. We will prove the first part. Since the dual of the dual is primal, the second part will follow.

Consider the set,

$$M = \{Z, u, v : \exists X, \text{ s.t.}, X \succeq Z, \forall i \ \ u_i = A_i \bullet X - b_i, v \leq C \bullet X\}$$

This can be thought of as the hypograph (the point below the graph) for the function $X, (\forall i \ \ A_i \bullet X - b_i), C \bullet X$ on first and last parts. Here Z is the symmetric matrix, u is a vector and v is a scalar. Since the function is concave (linear), so the hypograph is a convex set. Consider another convex set,

$$N = \{(0, 0, t) : t > p^*\}$$

Here p^* is the optimal value of primal (it is finite because $\mathcal{D} \neq \varnothing$). Hence, set M and N do not intersect (why?). By separating hyperplane theorem, there exist $\lambda_1, \lambda_2, \lambda_3, \alpha$, s.t.,

$$\lambda_1 \bullet Z + \lambda_2^T u + \lambda_3 v \leq \alpha \ \ \forall (Z, u, v) \in M \tag{2}$$

$$\lambda_1 \bullet 0 + \lambda_2^T 0 + \lambda_3 t \geq \alpha \ \ \forall (0, 0, t) \in N. \tag{3}$$

The left hand side of Eqn. 2 can only be upper bounded if $\lambda_1 \succeq 0$ and $\lambda_3 \geq 0$. The Eqn. 3 implies $\lambda_3 p^* \geq \alpha$. Assume $\lambda_3 > 0$, we will show this later by using the strict feasibility. Then,

$$\lambda_1' \bullet Z + \lambda_2'^T u + v \leq p^* \ \ \forall (Z, u, v) \in M. \tag{4}$$

Where $\lambda_1' = \frac{\lambda_1}{\lambda_3}$ and $\lambda_2' = \frac{\lambda_2}{\lambda_3}$. Using the definition of M, we get,

$$\lambda_1' \bullet X + \sum_i \lambda_{2,i}'(A_i \bullet X - b_i) + C \bullet X \qquad \leq p^* \ \ \forall X. \tag{5}$$

$$(\lambda_1' + \sum_i \lambda_{2,i}' A_i + C) \bullet X \leq p^* + b^T \lambda_2 \ \ \forall X \tag{6}$$

Here $\lambda_{2,i}'$ is the i^{th} entry of λ_2'. The Eqn. 6 implies $p^* + b^T \lambda_2 \geq 0$ and $\lambda_1' + \sum_i \lambda_{2,i}' A_i + C = 0$.

Exercise 3. Prove the last statement.

Substituting $\lambda_2' = -\lambda$, we get (remember that $\lambda_1' \succeq 0$),

$$p^* \geq b^T \lambda \text{ and } \sum_i \lambda_i A_i - C \in \mathcal{S}_n$$

So we get a feasible dual solution λ. Since dual is a minimization problem and from weak duality $p^* \leq d^*$,

$$p^* \geq b^T\lambda \geq d^* \Rightarrow p^* = d^*$$

Hence strong duality holds. The only part left to prove is that $\lambda_3 \neq 0$ using strict feasibility (notice that strict feasibility is not used till now).

Suppose $\lambda_3 = 0$, then Eqn. 2 and Eqn. 3 implies

$$\lambda_1 \bullet X + \sum_i \lambda_{2,i}(A_i \bullet X - b_i) \leq \alpha \leq \lambda_3 p^* = 0 \quad \forall X.$$

Take a strictly feasible X, then $A_i \bullet X - b_i = 0$. So $\lambda_1 \bullet X \leq 0 \Rightarrow \lambda_1 = 0$, since $\lambda_1 \succeq 0$. Then,

$$\sum_i \lambda_{2,i}(A_i \bullet X - b_i) \leq \alpha \leq \lambda_3 p^* = 0 \quad \forall X.$$

Lets assume that A_i's span the whole space for simplicity. For a strictly feasible X, left hand side is zero. Then if we move in some direction $X + \epsilon Y$, the value will be non-zero (otherwise $\lambda_2 = 0$ and the separating hyperplane is trivial). So in either that direction or the negative one the left hand side will be positive and will violate inequality. Hence $\lambda_3 \neq 0$.

$\square$

1.1 Counterexample for strong duality

From the last section we infer that strong duality holds for most of the cases in semidefinite programming. But still there exist cases when strong duality does not hold. Below, we give an example.

Consider the semidefinite programs for 3×3 matrices,

Primal

$$\begin{aligned} \max \quad & -x_{11} - x_{22} \\ \text{s.t.} \quad & x_{11} = 0, 2x_{13} + x_{22} = 1 \\ & X \succeq 0 \end{aligned}$$

Dual

$$\begin{aligned} & \min y_2 \\ \text{s.t.} \quad & \begin{pmatrix} y_1 + 1 & 0 & y_2 \\ 0 & y_2 + 1 & 0 \\ y_2 & 0 & 0 \end{pmatrix} \succeq 0 \end{aligned}$$

Exercise 4. Show that the above are semidefinite program and dual of each other.

For the primal problem, the first digonal entry forces x_{13} to be zero. Hence the optimal value is -1. For the dual problem, the last diagonal entry forces $y_2 = 0$. So the optimal value is 0. Hence, strong duality does not hold.

Exercise 5. Show that Slater's condition does not hold in this case.

Notice that the first constraint in primal is a weird way to say that first row and column are zero. Show that if we remove first row and column from the primal problem, then strong duality holds.