Lecture 15: Duality theory

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One of the strongest aspect of convex optimization is the presence of duality theory. The dual of a program is another convex optimization program which provides lot of information about the original program.

1 Dual of a linear program

We will motivate the duality theory by showing how to take the dual of a linear program. Suppose there is a linear program,

$$\begin{aligned} &\max 2x_1 + 3x_2 + x_3 \\ \text{s.t.} \quad & x_1 + x_2 + x_3 = 5 \\ & x_1 + 2x_2 = 10 \\ & x_1, x_2, x_3 \geq 0. \end{aligned}$$

What is the value of this program? On a close inspection, it will be clear that the objective function is a linear combination of constraints. Hence whatever be the feasible solution, the objective value will be 15.

Notice that it is required to have a feasible solution. In case there is no feasible solution then the objective value of the program will be $-\infty$. Remember that if a max optimization program does not have a feasible solution then its optimal value is $-\infty$ and if the min optimization program is infeasible then its optimal value is $+\infty$.

Lets take another example,

$$\begin{aligned} &\max 2x_1 + 3x_2 + x_3 \\ \text{s.t.} \quad & x_1 + x_2 + x_3 = 5 \\ & x_1 + 2x_2 + x_3 = 10 \\ & x_1, x_2, x_3 \geq 0. \end{aligned}$$

In this case it is not clear if 15 is the optimal value. But since $x_3 \geq 0$, the objective value is $15 - x_3$ and hence lesser than or equal to 15. So 15 is an upper bound on the objective value. Lets make this argument more precise. Suppose,

$$\begin{aligned} &\max c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0, \end{aligned}$$

is a linear program. If there exist a linear combination of constraints, say y, s.t., $y^T A = c$. Then $y^T b$ will be the optimal value of the program (given that there exist a feasible solution). But getting $y^T A = c$ every time is really improbable. Instead, if $y^T A \geq c$, then $y^T b$ is an upper bound on the value of the program. So we take the best upper bound.

$$\begin{aligned} &\min y^T b \\ \text{s.t.} \quad & y^T A \geq c. \end{aligned} \tag{1}$$

From the way this program is constructed, the optimal value of Eqn. 1 is always higher than optimal value of Eqn. 1. The Eqn. 1 is known as the *primal linear program.* The Eqn. 1 is known as the *dual linear program.*

The argument above showed that the dual objective value is higher than the primal objective value. This is known as the *weak duality*. In case of linear programs it is known that actually dual value is always same as the primal value if primal has a feasible solution. This is known as *strong duality*.

The terms primal and dual are interchangeable. The dual of the dual is the primal program. So if Eqn. 1 is considered as the primal program then the dual will be Eqn. 1.

One important thing to notice is that for every primal constraint there is a dual variable and for every primal variable there is a dual constraint. This relationship is much more stronger than it seems now.

Exercise 1. Show that the dual of the dual is the primal program (for the above primal and dual linear program). Hint: First convert the dual into primal form and then take the dual.

Exercise 2. Find the dual of the following linear program.

$$\begin{aligned} & \max c^T x \\ \text{s.t.} \quad & Ax \geq b \\ & x \leq 0, \end{aligned}$$

2 Duality theory

In this section we will see how the dual of a general cone program can be obtained. Suppose there is an optimization program,

$$\begin{aligned} & \max c^T x \\ \text{s.t.} \quad & Ax \leq b \\ x \in S. \quad & (S \text{ is some cone}) \end{aligned}$$

For this case, say we take the linear combination y or the rows of matrix A. Then we want $y^T b$ to be an upper bound on the value of $c^T x$ for any feasible x. One possible way is to satisfy $c^T x \leq y^T Ax \leq y^T b$. For the second inequality, y should be positive and for the first one $y^T A - c \in S^*$. Here S^* is the dual cone of S. So the dual program looks like,

$$\begin{aligned} & \min y^T b \\ \text{s.t.} \quad & y^T A - c \in S^* \\ & y \geq 0, \end{aligned}$$

Lets look at the two programs slightly differently.

$$\begin{aligned} & \max c^T x \\ \text{s.t.} \quad & Ax - b \leq 0 \Leftrightarrow Ax - b \in C \\ & x \in S. \quad (S, C \text{ are cones}) \end{aligned} \qquad\qquad \begin{aligned} & \min y^T b \\ \text{s.t.} \quad & y^T A - c \in S^* \\ & -y \in C^*, \end{aligned}$$

Cone C is the cone of all negative vectors, $\mathbb{R}^n_-$.

If the given program is a maximization problem, then if the primal variable is in cone S then the dual constraint is the membership in S^*. If the primal constraint is membership in cone C then the negative of dual variable is in cone C^*. For the minimization problem the relationship is opposite.

	Primal variable in S	Primal constraint in C
Max	Dual constraint in S^*	Negative of dual variable in C^*
Min	Negative of Dual constraint in S^*	Dual variable in C^*

Exercise 3. Try the previous exercise now.

3 Dual of a semidefinite program

Now it becomes easy to define the dual of a standard semidefinite program.

$$\begin{aligned} &\max C \bullet X \\ \text{s.t.}\quad &A_i \bullet X = b_i \quad \forall i \\ &X \in \mathcal{S}_n \end{aligned} \qquad\qquad \begin{aligned} &\min y^T b \\ \text{s.t.}\quad &\textstyle\sum_i y_i A_i - C \in \mathcal{S}_n \end{aligned}$$

The dual variable y is unconstrained because there is equality in the primal constraint. The dual program is in the second standard form discussed in the class previously.

Exercise 4. Find the dual of the following program.

$$\begin{aligned} &\min \tfrac{(b^T y)^2}{d^T y} \\ \text{s.t.}\quad &A^T y \le c. \end{aligned}$$