Lecture 14: Examples of semidefinite programming

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1 Fan's theorem

The sum of first k eigenvalues can also be written as a semidefinite program.

Theorem 1. *Fan: Given a symmetric matrix M and its eigenvalues $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_n$,*

$$\begin{aligned} \lambda_1 + \lambda_2 + \cdots \lambda_k &= \max M \circ X \\ & s.t. \;\; tr(X) = k \\ & I \succeq X \succeq 0 \end{aligned}$$

.

Proof. Suppose M has spectral decomposition $\lambda_1 x_1 x_1^T + \cdots + \lambda_n x_n x_n^T$. Then taking $X = x_1 x_1^T + \cdots + x_k x_k^T$ will give us value $\lambda_1 + \cdots + \lambda_k$. So now we need to show that any feasible X gives value less than $\lambda_1 + \cdots + \lambda_k$.

Since M is a symmetric matrix, by spectral decomposition it has complete basis of eigenvectors ($x_1, \cdots, x_n$ span the entire space).

Exercise 1. Show that M is diagonal in this basis with diagonal entries as eigenvalues.

Notice that the trace and positive semidefiniteness properties are preserved under change of basis.

Exercise 2. Show that $tr(AB) = tr(BA)$, hence $tr(U^T BU) = tr(B)$ for any orthogonal U. Also show that if $B \succeq 0$ then $U^T BU \succeq 0$ for any orthogonal U.

Lets look at the semidefinite program in the theorem in this eigenvector basis. Then $M \circ X = \sum_i X_{ii}\lambda_i$. From the constraints, $tr(X) = k$ and $I \succeq X \succeq 0$ (the identity matrix and zero matrix remain same under any basis transformation). Since $I - X \succeq 0$, all the entries $X_{ii} \geq 1$. Since $X \succeq 0$, implies $X_{ii} \geq 0$. Consider another optimization program (it is a linear program),

$$\begin{aligned} & \max \textstyle\sum_i z_i \lambda_i \\ \text{s.t.} \quad & \textstyle\sum_i z_i = k \\ & 1 \geq z_i \geq 0. \end{aligned}$$

By substituting $z_i = X_{ii}$ it is clear that the value of this program is at least the value of the semidefinite program in the theorem.

Exercise 3. Show that the maximum for this new program will occur on z's which have exactly k non-zero entries equal to 1.

So if we are allowed to choose k co-ordinates, s.t., $\sum_i \lambda_i z_i$ is maximum, then the best choice is the first k co-ordinates. Hence the maximum value of this linear program is $\lambda_1 + \cdots + \lambda_k$. This implies that the maximum value of semidefinite program in the theorem is $\lambda_1 + \cdots + \lambda_k$.

$\square$

2 Linear programs as a special case

Linear programming is a special case of semidefinite programs. It is obtained by considering the diagonal matrices in the standard form of semidefinite programming. Suppose the linear program is,

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & a_i^T x = b_i, \ \forall i = 1, \cdots, m \\ & x \geq 0. \end{aligned}$$

in the standard form. Say there is another semidefinite program,

$$\begin{aligned} \max \quad & C \circ X \\ \text{s.t.} \quad & A_i \circ X = b_i, \ \forall i = 1, \cdots, m \\ & X \succeq 0; \end{aligned}$$

Here C is the diagonal matrix with entries from c and A_i's are the diagonal matrices with diagonals a_i. Then the above mentioned two programs are actually equal. Given a solution of the linear program, it can be converted into a solution for semidefinite program by taking X to be the diagonal matrix with diagonal x. Similarly, if X is any solution for the semidefinite program, then $x = diag(X)$ will be a solution if linear program with the same objective value.

Hence any linear program can be converted into a semidefinite program by taking corresponding diagonal matrices for the constraints as well as the objective matrix.

3 Sum of squares

If a polynomial $p(x)$ can be written as a sum of squares then it is clearly positive for all values of x (here $x = (x_1, \cdots, x_n)$). In general, this gives a sufficient condition for positivity of the polynomial. To check whether a polynomial can be written as a sum of squares is a semidefinite programming feasibility problem.

To see this, first notice that a polynomial (of degree d) can be written as the dot product between two vectors $p(x) = p^T x_d$. Here x_d is the list of all degree $\leq d$ monomials. Then p is the vector of coefficients corresponding to those monomials. Now suppose $p(x) = \sum_i q_i(x)^2$, i.e., it can be written as the sum of squares. Say the degree of p is $2d$. Hence,

$$p(x) = \sum_i q_i(x)^2 = \sum_i x_d^T q_i^T q_i x_d = x_d^T (\sum_i q_i^T q_i) x_d$$

So, a polynomial is a sum of squares, iff, it can be written as $x_d^T Q x_d$ for some positive semidefinite Q. So, a polynomial is a sum of squares iff

$$\begin{aligned} p(x) &= x_d^T Q x_d \\ Q &\succeq 0. \end{aligned}$$

It might be confusing that why this is a semidefinite program. First the constraint $p(x) = x_d^T Q x_d$ is a linear constraint on the elements of Q. Also the absence of max/min might be confusing. This kind of problem without objective function and only constraint is called a semidefinite programming feasibility problem. It is a special case of semidefinite programming (why ??).

This semidefinite program is important in giving bounds on the minimum value of a polynomial. Consider,

$$\begin{aligned} & \max \lambda \\ \text{s.t.} \quad & p(x) - \lambda = x_d^T Q x_d \\ & Q \succeq 0. \end{aligned}$$

The value of this program (say s^*) satisfies $p(x) \geq s^*$ for all x. This might not be the biggest s, s.t. $p(x) \geq s, \ \forall x$; since the representation as sum of squares is only a sufficient condition for positivity. Though if $n = 1$, this gives us the tight bound.

Exercise 4. Prove that for a single variate polynomial p, it is positive iff it can be written as a sum of squares (Hint: Look at the factorization of p in complex domain).