Lecture 12: Positive semidefinite cone

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Positive semidefinite matrices are symmetric matrices whose eigenvalues are non-negative. They can also be thought of as the gram matrix of a set of vectors. Today's lecture will look at their special properties and the cone generated by them.

1 Properties of semidefinite matrices

1.1 Principal submatrix

A principal submatrix P of a matrix M is obtained by selecting a subset of rows and the same subset of columns. If M is positive semidefinite then all its principal submatrices are also positive semidefinite.

This follows by considering the quadratic form $x^T Mx$ and looking at the components of x corresponding to the defining subset of principal submatrix. The converse is trivially true.

Exercise 1. Show that the determinant of a positive semidefinite matrix is non-negative. Hence, show that all the principal minors are non-negative. Actually the converse also holds true, i.e., if all the principal minors are non-negative then the matrix is positive semidefinite.

1.2 Diagonal elements

If the matrix is positive semidefinite then its diagonal elements should *dominate* the non-diagonal elements. The quadratic form for M is,

$$x^T Mx = \sum_{i,j} M_{i,j} x_i x_j. \tag{1}$$

Here x_i's are the respective components of x. If M is positive semidefinite then Eqn. 1 should be non-negative for every choice of x.

By choosing x to be a standard basis vector e_i, we get $M_{ii} \geq 0, \ \forall i$. Hence, all diagonal elements are nonnegative and $tr(M) \geq 0$. If x is chosen to have only two nonzero entries, lets say at i and j position, then Eqn. 1 implies,

$$M_{i,j} \leq \sqrt{M_{ii} M_{jj}} \leq \frac{M_{ii} + M_{jj}}{2}.$$

Where the second inequality follows from AM-GM inequality. This shows that any off diagonal element is less than the diagonal element in its row or in its column.

2 Schur's complement

Given a 2×2 block matrix,

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

the *schur complement* of the matrix D in M is $A - BD^{-1}C$. This gives a criteria to decide if a 2×2 symmetric block matrix is positive definite or not.

Theorem 1. *Suppose M is a symmetric 2×2 block matrix,*

$$M = \begin{pmatrix} A & B \\ B^T & D \end{pmatrix}.$$

It is positive definite iff D and the schur complement $A - BD^{-1}B^T$, both are positive definite.

Proof. Notice that,

$$M = \begin{pmatrix} I & BD^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} A - BD^{-1}B^T & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} I & BD^{-1} \\ 0 & I \end{pmatrix}^T \tag{2}$$

It is known that,

$$\begin{pmatrix} I & BD^{-1} \\ 0 & I \end{pmatrix}^{-1} = \begin{pmatrix} I & -BD^{-1} \\ 0 & I \end{pmatrix}.$$

Hence $M = P^T N P$ where P is invertible and N is a diagonal matrix. So M is positive definite if and only if N is positive definite. It is easy to check when a block diagonal matrix is positive definite. That exactly gives us the that D and the schur complement $A - BD^{-1}B^T$ both have to be positive definite. □

Exercise 2. Given a matrix,

$$M = \begin{pmatrix} I & B \\ B^T & I \end{pmatrix},$$

and the condition $B = B^T$. Show that it is positive definite iff $I \pm B \succ 0$.

3 Positive semidefinite cone

Consider the vector space of symmetric $n \times n$ matrices, $\mathbb{R}^{\frac{n(n+1)}{2}}$. We focus on the set of positive semidefinite matrices in this space.

It was seen that if M, N are positive semidefinite, then $\alpha M + \beta N$ is also positive semidefinite for positive α, β. Hence, the set of positive semidefinite matrices is a convex cone in $\mathbb{R}^{\frac{n(n+1)}{2}}$. The cone is denoted $\mathcal{S}_n$.

If $M \succeq 0$ then $-M$ is not positive semi-definite. So the cone $\mathcal{S}_n$ does not contain a line. If we look at the positive definite matrices. They form the interior of the cone. To prove this, we show that for any positive definite matrix M, there exist a ball of size ϵ centered at M and contained in the cone of positive semidefinite matrices.

Theorem 2. *If given an $n \times n$ matrix $M \succ 0$ (positive definite). Then, $M - \epsilon N \succeq 0$ for small enough ϵ and all symmetric N whose norm is 1.*

Proof. The norm of N is 1, i.e., $N \bullet N = 1$. So every entry of N is at-most 1 (Exercise: Why ?). For every unit vector v, every element is bounded by 1 too. So $v^T N v = \sum_{i,j} v_i v_j N_{ij} \leq n^2$.

Exercise 3. The choice $\epsilon = \frac{\lambda_n}{2n^2}$ will work, where λ_n is the least eigenvalue of M.

□

Identity is positive definite, so interior is not empty.

Notice that we did not take the space of all $n \times n$ matrix. In that case the set of positive semidefinite matrices will not be solid.

Hence, $\mathcal{S}_n$ is a convex cone that does not contain a line and has non-empty interior. This implies that the cone $\mathcal{S}_n$ is proper. Define the generalized inequality with respect to this cone.

$$M \succeq N \Leftrightarrow M - N \succeq 0$$

The positive semidefinite cone is generated by all rank one matrices xx^T. They form the extreme rays of the cone. The positive definite matrices lie in the interior of the cone. The positive semidefinite matrices with at least one eigenvalue zero are on the boundary.

3.1 Self dual cone

The inner product in this space is the $\bullet$ operation between matrices.

$$A \bullet B = \sum_{i,j} A_{ij} B_{ij} = tr(A^T B)$$

The dual cone of $\mathcal{S}_n$ is the cone $\mathcal{S}'_n$, s.t.,

$$\mathcal{S}'_n = \{M : M \bullet N \geq 0,\ \forall N \in \mathcal{S}_n\}$$

Consider the hadamard product of two positive semidefinite matrices. It was proved in the last class that $M \circ N \succeq 0$.

Exercise 4. If $M \circ N \succeq 0$, prove that $M \bullet N \geq 0$.

Hence every positive semidefinite matrix is part of the dual cone $\mathcal{S}'_n$. It implies

$$\mathcal{S}_n \subseteq \mathcal{S}'_n. \tag{3}$$

Now consider a symmetric matrix $M \notin \mathcal{S}_n$. There exist at least one negative eigenvalue λ and an eigenvector v corresponding to it. So,

$$0 > \lambda = v^T M v = M \bullet (vv^T).$$

This implies that $M \notin \mathcal{S}'_n$. Hence,

$$\mathcal{S}'_n \subseteq \mathcal{S}_n. \tag{4}$$

From Eqn. 3 and Eqn. 4, we get $\mathcal{S}'_n = \mathcal{S}_n$. The cone of positive semidefinite cone is a self dual cone.