Lecture 11: Positive semidefinite matrix

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In the last lecture a positive semidefinite matrix was defined as a symmetric matrix with non-negative eigenvalues. The original definition is that a matrix $M \in L(V)$ is positive semidefinite iff,

1. M is symmetric,
2. $v^T M v \geq 0$ for all $v \in V$.

If the matrix is symmetric and

$$v^T M v > 0, \ \forall v \in V,$$

then it is called positive definite. When the matrix satisfies opposite inequality it is called negative definite.

The two definitions for positive semidefinite matrix turn out be equivalent. In the next section, we identify many different definitions with positive semidefinite matrices.

1 Equivalent definitions of positive semidefinite matrices

Theorem 1. *For a symmetric $n \times n$ matrix $M \in L(V)$, following are equivalent.*

1. *$v^T M v \geq 0$ for all $v \in V$.*
2. *All the eigenvalues are non-negative.*
3. *There exist a matrix B, s.t., $B^T B = M$.*
4. *Gram matrix of vectors $u_1, \cdots, u_n \in U$, where U is some vector space. Hence*

$$\forall i, j; \ M_{i,j} = v_i^T v_j.$$

Proof. 1 $\Rightarrow$ 2: Say λ is an eigenvalue of M. Then there exist eigenvector $v \in V$, s.t., $Mv = \lambda v$. So $0 \leq v^T M v = \lambda v^T v$. Since $v^T v$ is positive for all v, implies λ is non-negative.

2 $\Rightarrow$ 3: Since the matrix M is symmetric, it has a spectral decomposition.

$$M = \sum_i \lambda_i x_i x_i^T$$

Define $y_i = \sqrt{\lambda_i} x_i$. This definition is possible because λ_i's are non-negative. Then,

$$M = \sum_i y_i y_i^T.$$

Define B to be the matrix whose columns are y_i. Then it is clear that $B^T B = M$. From this construction, B's columns are orthogonal. In general, any matrix of the form $B^T B$ is positive semi-definite. The matrix B need not have orthogonal columns (it can even be rectangular).

But this representation is not unique and there always exists a matrix B with orthogonal columns for M, s.t., $B^T B = M$. This decomposition is unique if B is positive semidefinite. The positive semidefinite B, s.t., $B^T B = M$, is called the square root of M.

Exercise 1. Prove that the square root of a matrix is unique.

Hint: Use the spectral decomposition to find one of the square root. Suppose A is any square root of M. Then use the spectral decomposition of A and show the square root is unique (remember the decomposition to eigenspaces is unique) .

3 $\Rightarrow$ 4: We are given a matrix B, s.t., $B^T B = M$. Say the rows of B are $u_1, \cdots, u_n$. Then, from the definition of matrix multiplication,

$$\forall i, j; \ M_{i,j} = v_i^T v_j$$

Exercise 2. Show that for a positive semidefinite matrix $M \in L(V)$, there exists $v_1, \cdots, v_n \in V$, s.t, M is a gram matrix of $v_1 \cdots, v_n$.

$4 \Rightarrow 1$: Suppose M is the gram matrix of vectors $u_1, \cdots, u_n$. Then,

$$x^T M x = \sum_{i,j} M_{i,j} x_i x_j = \sum_{i,j} x_i x_j (v_i^T v_j),$$

where x_i is the i^{th} element of vector x. Define $y = \sum_i x_i v_i$, then,

$$0 \geq y^T y = \sum_{i,j} x_i x_j (v_i^T v_j) = x^T M x.$$

Hence $x^T M x \geq 0$ for all x.

Exercise 3. Prove that $2 \Rightarrow 1$ and $3 \Rightarrow 1$ directly.

□

Remark: A matrix M of the form $M = \sum_i x_i x_i^T$ is positive semidefinite (Exercise: Prove it), even if x_i's are not orthogonal to each other.

Remark: A matrix of the form yx^T is a rank one matrix. It is rank one because all columns are scalar multiples of y. Similarly, all rank one matrices can be expressed in this form.

Exercise 4. A rank one matrix yx^T is positive semi-definite iff y is a positive scalar multiple of x.

2 Some examples

- An $n \times n$ identity matrix is positive semidefinite. It has rank n. All the eigenvalues are 1 and every vector is an eigenvector. It is the only matrix with all eigenvalues 1 (Prove it).
- The all 1's matrix J $(n \times n)$ is a rank one positive semidefinite matrix. It has one eigenvalue n and rest are zero.
- The matrix

$$M = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix},$$

  is positive semidefinite. Because, the quadratic form $x^T M x = (x_1 - x_2)^2$, where x_1, x_2 are two components of x.
- Suppose any symmetric matrix M has maximum eigenvalue λ. The matrix $\lambda' I - M$, where $\lambda' \geq \lambda$ is positive semidefinite.

3 Composition of semidefinite matrices

- The direct sum matrix $A \oplus B$,

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix},$$

  is positive semidefinite iff A and B both are positive semidefinite. This can most easily be seen by looking at the quadratic form $x^T (A \oplus B) x$. Divide x into x_1 and x_2 of the required dimensions, then

$$x^T (A \oplus B) x = x_1^T A x_1 + x_2^T B x_2.$$

– The tensor product $A \otimes B$ is positive semidefinite iff A and B are both positive semidefinite or both are negative semidefinite. This follows from the fact that given the eigenvalues $\lambda_1, \cdots, \lambda_n$ for A and $\mu_1, \cdots, \mu_m$ for B; the eigenvalues of $A \otimes B$ are

$$\forall i, j, \ \lambda_i \mu_j.$$

– The sum of two positive semidefinite matrices is positive semidefinite.
– The product of two positive semidefinite matrices need not be positive semidefinite.

Exercise 5. Give an example of two positive semidefinite matrices whose product is not positive semidefinite.

– The hadamard product of two positive semidefinite matrices A and B, $A \circ B$, is also positive semidefinite. Since A and B are positive semidefinite for some vectors $u_1, \cdots, u_n$ and $v_1, \cdots v_n$. The hadamard matrix will be the gram matrix of $u_i \otimes v_i$'s. Hence it will be positive semidefinite.
– The inverse of a *positive definite* matrix is positive definite. The eigenvalues of the inverse are inverses of the eigenvalues.
– The matrix $P^T M P$ is positive semidefinite if M is positive semidefinite.