Lecture 10: Spectral decomposition

Rajat Mittal *

IIT Kanpur

1 Spectral decomposition

In general, a square matrix M need not have all the n eigenvalues. Some of the roots of $det(\lambda I - M)$ might be complex. The eigenvectors corresponding to different eigenvalues need not be orthogonal.

A *normal matrix* is defined to be a matrix M, s.t., $MM^T = M^T M$. The eigenspaces corresponding to these matrices are orthogonal to each other, though the eigenvalues can still be complex.

Theorem 1. *Spectral theorem: For a normal matrix $M \in L(V)$, there exists an orthonormal basis $x_1, \cdots, x_k$ of V, s.t.,*

$$M = \sum_{i=1}^{n} \lambda_i x_i x_i^*.$$

Here $\forall i\ \lambda_i \in \mathbb{C}. x_i^*$ is the adjoint of x_i.

Exercise 1. Show that x_i is an eigenvector of M with eigenvalue λ_i.

Remark: Notice that $y^* x$ is a scalar, but yx^* is a matrix.

Remark: The λ_i's need not be different. If we collect all the x_i's corresponding to a particular eigenvalue λ, the space spanned by those x_i's is the eigenspace of λ.

Proof. Idea: First it will be shown that given an eigenspace S (say corresponding to eigenvalue λ), the matrix M acts on the space S and $S^\perp$ separately. That is $Mv \in S$ if $v \in S$ and $Mv \in S^\perp$ if $v \in S^\perp$. This implies that M is a linear operator on $S^\perp$.

Since S is an eigenspace, $Mv \in S$ if $v \in S$. For a vector $v \in S$,

$$MM^T v = M^T M v = \lambda M^T v.$$

This shows that M^T preserves the subspace S. Suppose $v_1 \in S^\perp,\ v_2 \in S$, then $M^T v_2 \in S$. So,

$$0 = v_1^T (M^T v_2) = (M v_1)^T v_2.$$

Hence $Mv_1 \in S^\perp$. Hence, matrix M acts separately on S and $S^\perp$.

From the fundamental theorem of Algebra, there is at least one root of $det(\lambda I - M) = 0$. Start with the eigenspace of that root (Exercise: Show that it is not empty). From the previous paragraph we can restrict the matrix to orthogonal subspace and find another root. Using induction, we can divide the entire space into orthogonal eigenspaces.

Exercise 2. Show that if we take the orthonormal basis of all these eigenspaces, then we get the required decomposition.

□

Clearly the spectral decomposition is not unique (essentially because of the multiplicity of eigenvalues). But the eigenspaces corresponding to each eigenvalue are fixed. So there is a unique decomposition in terms of eigenspaces and then any orthonormal basis of these eigenspaces can be chosen.

Remark: It is also true that an eigenvalue is a root of characteristic polynomial with multiplicity k, then its eigenspace is of dimension k. The eigenvalues and eigenvectors have more structure if we look at specific classes of normal matrices.

* Thanks to John Watrous's course notes, IQC, Waterloo.

1.1 Symmetric matrix

A matrix is symmetric if $M = M^T$. It is clearly normal. All the roots of the characteristic polynomial are real and hence all eigenvalues are real. The eigenvectors can be complex or real. But an orthonormal basis of real eigenvectors can always be chosen.

This statement is true for a more general class of matrices called *hermitian matrices* (analog of symmetric in complex domain).

Conversely if all the eigenvalues are real and there exist a real orthonormal basis of eigenvectors then the matrix is symmetric (from Spectral theorem). A matrix of the form $B^T B$ for any matrix B is always symmetric.

The sum of two symmetric matrices is symmetric. But the multiplication of two symmetric matrices need not be symmetric.

Exercise 3. Give an example of two symmetric matrices whose multiplication is not symmetric.

1.2 Orthogonal matrix

A matrix M is orthogonal if $MM^T = M^T M = I$. In other words, the columns of M form an orthonormal basis of the whole space. Orthogonal matrices need not be symmetric, so their eigenvalues can be complex. For an orthogonal matrix $M^{-1} = M^T$.

Orthogonal matrices can be viewed as matrices which do change of basis. Hence they preserve the angle (inner product) between the vectors. So for orthogonal M,

$$u^T v = (Mu)^T Mv.$$

Exercise 4. Prove that the absolute value of the eigenvalues of an orthogonal matrix is 1.

If two matrices A, B are related by $A = M^{-1}BM$, then they are unitarily equivalent. Unitary matrices are analog of orthogonal matrices in complex domain. If two matrices are unitarily equivalent then they are similar.

Spectral theorem can be stated as the fact that normal matrices are unitarily equivalent to a diagonal matrix. The diagonal of the diagonal matrix contains the eigenvalues.

1.3 Positive semidefinite matrix

A matrix M is positive semidefinite if it is symmetric and all its eigenvalues are non-negative. This class is going to be one of the most important class of matrices in this course. If all eigenvalues are strictly positive then it is called a positive definite matrix.

2 Singular value decomposition

Singular value decomposition is one of the most important factorizations of a matrix. The statement says,

Theorem 2. *Given a linear operator M in $L(V, W)$. There exists a decomposition of the form:*

$$M = \sum_{i=1}^{r} s_i y_i x_i^T$$

Where $x_1, \cdots, x_r$ (called right singular vectors) and $y_1, \cdots, y_r$ (called left singular vectors) are orthonormal basis of V and W respectively. The numbers $s_1, \cdots, s_r$ (called singular values) are positive real numbers and r itself is the rank of the matrix M.

Remark: Note that this statement is easy to prove if we don't need any condition on y_i's. Any basis of V will be sufficient to construct such a decomposition (why?). We can even choose all singular values to be 1 in that case. But it turns out that with the singular values we can make the y_i's to be orthonormal.

The statement of the theorem can also be written as $M = A\Delta B^*$, where $A \in L(W), B \in L(V)$ are orthogonal matrices and Δ is the diagonal matrix of singular values. With this interpretation, any linear operation can be viewed as rotation in subspace V then scaling the standard basis and then another rotation in W subspace.

The proof of singular value decomposition follows by applying spectral decomposition on matrices MM^T and M^TM. The eigenvectors of MM^T are left singular vectors and eigenvectors of M^TM are right singular vectors of M. The eigenvalues of MM^T or M^TM are the singular values of M.

Exercise 5. Prove that M^TM and MM^T have the same set of eigenvalues (hint: use singular value decomposition).

3 Alternate characterization of eigenvalues of a symmetric matrix

The eigenvalues of a symmetric matrix $M \in L(V)$ $(n \times n)$ are real. So they can be arranged in the order,

$$\lambda_1 \geq \cdots \geq \lambda_n.$$

By spectral theorem, the eigenvectors form an orthonormal basis. Say the eigenvectors are $v_1, \cdots, v_n$, where v_i is the eigenvector with eigenvalue λ_i. Any vector $v \in V$ with length 1 can be written as,

$$v = \sum_i \theta_i v_i, \ \sum_i \theta_i^2 = 1.$$

This implies that the quadratic form v^TMv is,

$$v^TMv = \sum_i \lambda_i \theta_i^2.$$

Hence the maximum value of quadratic form v^TMv is λ_i when $v = v_i$. So, the maximum eigenvalue λ_i is the maximum value of the quadratic form v^TMv. The eigenvector v_1 is the vector which maximizes the quadratic form.

The same argument tells us that λ_2 is the maximum value achieved by v^TMv for all v orthogonal to v_1. Generalizing, λ_i is the maximum value achieved by v^TMv, when v is orthogonal to $v_1, \cdots, v_{i-1}$.