Lecture 4: Partially Ordered Sets

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We will learn about relations over sets in this lecture. Relations are *easier to manage* if they satisfy certain properties like being reflexive, transitive, symmetric or antisymmetric. We will see the definitions of these properties, with special emphasis to *equivalence relations*.

Some of these relations give rise to partial or total orders in sets. The aim of these lecture notes is to look at these *posets* (partially ordered sets) and their properties.

1 Relations

A relation R from a set S to a set T can be thought of as a subset of the set $R \subseteq S \times T$ (set of all pairs), where

$$S \times T = \{(a, b) | \ a \in S, \ b \in T\}.$$

In other words, it can be thought of as a function from $S \times T$ to $\{0, 1\}$. Here, if a pair (x, y) is in a relation then it maps to 1 otherwise 0. There are many examples like less/greater/equal (between numbers), divides (between numbers), friends (between people) and contained (between sets).

Exercise 1. Show that a function is a relation with some other restrictions. What are those restrictions?

We will mostly be interested in relations from S to S itself in this course. As mentioned before, there are certain properties of a relation which gives it more structure and easier to understand. Let us look at the definition of some of these properties.

- Reflexive: A relation R is called *reflexive* if for all $s \in S$, we have $(s, s) \in R$.
- Symmetric: A relation R is called *symmetric* if $(s, t) \in R$ implies $(t, s) \in R$.
- Antisymmetric: A relation R is called *antisymmetric* if $(s, t) \in R$ and $(t, s) \in R$ implies $s = t$.

  Exercise 2. True or False: A relation is either symmetric or antisymmetric.
- Transitive: A relation R is called *transitive* if $(s, t) \in R$ and $(t, u) \in R$ implies $(s, u) \in R$.

Exercise 3. For every subset of the properties mentioned above, construct a relation which satisfies only those properties. Give a proof if no such relation exists.

Exercise 4. What properties does the relation *equality* satisfy?

1.1 Equivalence relations

Equivalence relations form one of the most useful set of relations which you will encounter very frequently. A relation R is called an *equivalence relation* if it is reflexive, symmetric and transitive.

Exercise 5. Give a natural example of an equivalence relation.

Exercise 6. Show that the congruence relation $a = b \mod m$ is an equivalence relation over integers.

The most important aspect of equivalence relation is that it partitions the whole set into *equivalence classes*. For any $s \in S$, let $E(s)$ be the set of all elements which are related to s,

$$E(s) = \{t | \ (s, t) \in R\}.$$

We call $E(s)$ to be the equivalence class of s.

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Exercise 7. When will $E(s)$ be equal to $E(t)$?

Since the relation R is reflexive, we know that $t \in E(t)$. So, if $E(s) = E(t)$ then $t \in E(s)$, which implies $(s,t) \in R$.

Consider the case when $(s,t) \in R$. If $u \in E(t) \Rightarrow (t,u) \in R$, then by transitivity $(s,u) \in R \Rightarrow u \in E(s)$. By a similar argument $u \in E(s) \Rightarrow u \in E(t)$.

To conclude, if R is equivalence relation then $E(s) = E(t)$ if and only if $(s,t) \in R$. You can prove another fact about equivalence classes,

Exercise 8. If s and t are not related, then $E(s) \cap E(t) = \emptyset$.

Combining the two facts, two equivalence classes are either identical (if the elements are related) or completely disjoint (if the elements are not related). In other words, in a particular equivalence class, an element is related to every element of that class and no-one else.

This means, we can define equivalence classes without the help of the generating element. Given an equivalence relation, just group the elements which are related to each other. These equivalence classes will partition the whole set. Remember, a partition of a set is a collection of disjoint subsets whose union is the entire set.

Similarly, given a partition, we can create an equivalence relation. Every element is related and only related to elements in the same partition. This gives us the following theorem.

Theorem 1. *The equivalence classes of an equivalence relation R partition the whole set. Conversely, given a partition, we can define an equivalence relation R whose equivalence classes are the partitions.*

Exercise 9. Color the four vertices of a square either red or blue. Let two colorings be related if one can be obtained from other through rotation, reflection or rotation then reflection. Show that it is an equivalence relation. What are the equivalence classes here?

Intuitively, equivalence relations generalize the concept of equality (in some sense). We will see lot of equivalence relations throughout this course.

2 Partial order

We took care of equality in the last subsection, let us move on to inequalities.

Exercise 10. What properties does the relation *greater than or equal to* satisfy?

A relation R is called a *partial order* if it is reflexive, transitive and antisymmetric. Notice the difference from equivalence relation, we switch symmetric to antisymmetric. A relation is called a *total order* if it is a partial order and for every pair $s,t \in S$ either $(s,t) \in R$ or $(t,s) \in R$. Notice that inequality is a total order over integers but partial order over complex numbers.

Exercise 11. Show that the power set of $\{1, 2, \cdots, n\}$ forms a poset with relation $\subseteq$ (A is related to B iff $A \subseteq B$).

It is convenient to represent a partial order (or total order) by the notation $x \leq y$ instead of $(x,y) \in R$. We will say $x < y$ if $x \leq y$ and $x \neq y$.

Exercise 12. What is the number of total orders on a set S of size n?

$n!$. Some people call total order a *linear order* too.

Exercise 13. Define $a, b \in \mathbb{C}$ to be related if absolute value of a is less than the absolute value of b. Is this a total order?

A set with a partial order is called a *poset*. Again, we frequently encounter posets in computer science.

2.1 Posets

We look at a convenient way to represent a poset known as *Hasse Diagram*.

Suppose R is a partial order. Connect $x, y \in S$ by a line if $x < y$ and there does not exist any z, s.t., $x < z < y$.

Exercise 14. Given x, show that y is unique when R is a total order.

In other words y is *the* least element larger than x for a total order, then we connect y with x. A total order will be represented by a straight line this way and is basically a permutation.

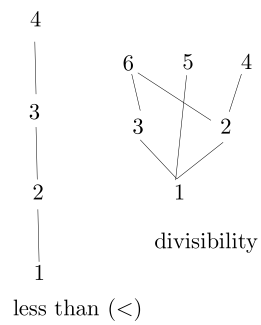

Fig. 1. Hasse diagram, first one is a total order.

If we draw similar diagram for a partial order with y above x (the upper element is larger than the smaller element), it is called the Hasse diagram for the poset. Some examples of Hasse diagram are given in Fig. 1.

A complete description of a relation can be obtained by its Hasse diagram with transitivity and reflexivity of the relation. Basically, Hasse diagram removes the edges of relation which can be obtained by transitivity and reflexivity.

An element x is called a *maximal (respectively minimal)* element of a poset if no element appears on top (bottom) of x and is connected to it. That means, x is maximal if and only if there does not exist y such that $x < y$. Notice that the maximal (minimal) element need not be unique.

2.2 Chains and anti-chains

Let us say that S is a poset with partial order $\leq$. If we see a vertical connected line in a Hasse diagram of S, it signifies a subset of the poset which has total order with respect to the relation $\leq$.

Such subsets are called *chains*. Formally, a chain C is a subset of S such that any two elements are comparable. On the other hand, a subset such that no two elements are comparable is called an *anti-chain*.

Exercise 15. What is the longest chain in the power set of $\{1, 2, \cdots, n\}$ with $\subseteq$ as the relation.

You can similarly ask, What is the longest anti-chain of this poset (power set with inclusivity as relation)? An anti-chain in this poset is called a *Sperner Family*. A simple observation is, all sets of same cardinality form an anti-chain. This implies that an easy lower bound of $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ can be given on the longest anti-chain. The theorem below shows that this is the best you can do.

Theorem 2 (Sperner's theorem). *Consider the poset formed by the power set of $[n] = \{1, 2, \cdots, n\}$ with relation of inclusivity. The longest anti-chain has size $\binom{n}{\lfloor \frac{n}{2} \rfloor}$. The longest chain either consists of all subsets of size $\lfloor \frac{n}{2} \rfloor$ or $\lceil \frac{n}{2} \rceil$.*

Note 1. There is a unique anti-chain when n is even of subsets of size $\frac{n}{2}$.

Proof. Let us define a maximal chain (m-chain) to be,

$$\emptyset = S_0 \subseteq S_1 \subseteq S_2 \cdots S_n = [n].$$

Exercise 16. Show that there are total $n!$ m-chains, one for every permutation.

By the definition of chain and anti-chain, there can't be two elements common between a chain and an anti-chain. This holds for the special case of m-chains too.

The idea of the proof is to show that any set S in A appears in *lot* of m-chains (say s). Since no chain can have two elements of A, the number of elements of A are bounded by $\frac{n!}{s}$.

Let us count the number of m-chains containing a set $S \subseteq [n]$ of size k. We can choose the first k elements of the m-chain in $k!$ ways (permutations of elements of S). The rest of the elements can be chosen in $(n-k)!$ ways (permutations of elements of $[n] - S$). So, set S is contained in $k!(n-k)!$ m-chains.

Exercise 17. Show that every set S is contained in at least $\frac{n!}{\binom{n}{\lfloor \frac{n}{2} \rfloor}}$ m-chains.

Show that $k!(n-k)!$ is minimum for $k = \lceil \frac{n}{2} \rceil$.

Since the total number of m-chains are $n!$ and no two elements of A are contained in a single m-chain. We get that there are at most $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ elements in A.

From the proof, it is clear that $|A| = \binom{n}{\lfloor \frac{n}{2} \rfloor}$ if and only if all sets have cardinality $\lfloor \frac{n}{2} \rfloor$ or $\lceil \frac{n}{2} \rceil$. When n is even, this two cardinalities are equal and hence every subset (element of A) needs to be of size $\frac{n}{2}$.

When n is odd, it is possible that some sets are of size $k := \lfloor \frac{n}{2} \rfloor$ and some of $k+1 = \lceil \frac{n}{2} \rceil$. To complete the proof, we will show the following lemma.

Lemma 1. *If a longest anti-chain contains a subset of size k then it will contain all subsets of size k.*

Exercise 18. Why is it enough?

Proof of lemma. Let S be a subset of size k in A. From S, we can reach any k size subset S' by a sequence of operations, where every operation consists of adding one and removing one element.

In other words, there is a sequence,

$$S = S_0 \subset T_1 \supset S_1 \cdots T_l \supset S_l = S',$$

where each S_i is of size k and each T_i is of size $k+1$.

Since S_0 is contained in A, T_1 can't be. Since A is the longest anti-chain, every chain has at least one element in A. So, if T_1 is not in A, implies S_1 is in A.

Repeating this argument, every S_i is in A and no T_i is in A. So, S' is in A. □

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The main idea of the proof was that the intersection of a chain and an anti-chain can have at most one element.

This shows that if there is a chain of r elements then the poset can't be partitioned with less than r anti-chains. Similarly, if there is an anti-chain of size r then the poset can't be partitioned with less than r chains.

Actually, much stronger results hold. If the longest chain is of size r, then there exists a partition with r anti-chains. Similarly, if the longest anti-chain is of size r, then there exists a partition with r chains (Dilworth's theorem).

We will not prove these theorems here. Interested students are encouraged to read about these results.

References

1. K. H. Rosen. Discrete Mathematics and Its Applications. *McGraw-Hill*, 1999.
2. N. L. Biggs. Discrete Mathematics. *Oxford University Press*, 2003.