Planarizing Gadgets for Perfect Matching Do not Exist*

Rohit Gurjar[1]**, Arpita Korwar[1], Jochen Messner[2],
Simon Straub[3], and Thomas Thierauf[2]

[1] IIT Kanpur, India
[2] Aalen University, Germany
[3] University of Ulm, Germany

Abstract. To reduce a graph problem to its planar version, a standard technique is to replace crossings in a drawing of the input graph by planarizing gadgets. We show unconditionally that such a reduction is not possible for the perfect matching problem and also extend this to some other problems related to perfect matching. We further show that there is no planarizing gadget for the Hamiltonian cycle problem.

1 Introduction

The *perfect matching problem* is a very fundamental computational problem (see, e.g., [17, 23]). Edmonds [8] developed a polynomial-time algorithm, but still it is unknown whether there is an efficient *parallel* algorithm for the perfect matching problem, i.e., whether it is in NC. In their seminal result, Mulmuley, Vazirani, and Vazirani [27] isolated a perfect matching by assigning random weights to the edges. This yielded a randomized parallel algorithm for the problem, it is in RNC. A derandomization of this algorithm is a challenging open problem.

There are NC algorithms for the perfect matching problem for special graph classes, for example for regular bipartite graphs [22], dense graphs [3], and strongly chordal graphs [4].

Here we consider *planar graphs*. Planarity is an interesting property with respect to the perfect matching problems, and seems to change the complexity of the problem drastically:

- Valiant [29] showed that counting the number of perfect matchings in a graph is a hard problem, namely it is #P-complete,
- whereas for planar graphs, Kasteleyn [18] showed that a Pfaffian orientation can be computed in polynomial time, which leads to a polynomial time algorithm for counting the number of perfect matchings. Vazirani [30] showed that the problem is in fact in NC.

* Work supported in part by the Indo-German DST-DFG program, DFG grant TH 472/4-1 and DST grant DST/CS/20100251.
** Partially supported by TCS Research Fellowship.

In contrast, no NC algorithm is known for the *construction* of a perfect matching in planar graphs. This is a puzzling state of affairs because, intuitively, counting seems to be a harder problem than construction. There is, however, an RNC algorithm for the construction problem [27].

Much work has been done on the perfect matching for *bipartite planar* graphs [26, 24, 21, 6, 13, 7]. The current best bound on the problem is unambiguous logspace, UL, for decision and construction [7]. Note that for the bipartite perfect matching problem no better bounds are known than for the general perfect matching problem.

In this paper, we investigate the question of whether there is a logspace or NC reduction from the perfect matching problem to the planar perfect matching problem. It is quite possible that such a reduction exists.

- Such a reduction would be a breakthrough result because it would derandomize the RNC algorithm for perfect matching. Many researchers conjecture that such a derandomization is possible (see, e.g., [1]). Hence, this could be one way of doing it.
- A reduction does not necessarily maintain the number of perfect matchings. Hence, it does not imply an unexpected collapse of complexity classes.

An obvious approach to such a reduction is a *planarizing gadget*: a planar graph that locally replaces the crossing edges of a given drawing of a graph. It is natural to suspect that any more globally acting reduction would be very involved to construct. Examples for planarizing gadgets are the reductions of 3-colorability and vertex cover to their planar versions [10]. In contrast, because of the four color theorem, a planarizing gadget for k-colorability cannot exist for $k \geq 4$. Datta et al. [5] have recently used a planarizing gadget to investigate the complexity of computing the determinant of a matrix, which is the adjacency matrix of a planar graph. They construct a gadget that reduces the general determinant to the planar determinant. Therefore, both problems have the same complexity, they are GapL-complete. The analogous result has been shown for the permanent, again via some planarizing gadget. Therefore, the permanent and the planar permanent are #P-complete.

Our first result is to construct an obstacle in getting an NC algorithm for the perfect matching problem: we show that planarizing gadgets for perfect matching do not exist. We extend the result to unique perfect matching, weighted perfect matching, exact perfect matching, and counting modulo k perfect matching.

The planar Hamiltonian cycle problem was shown to be NP-complete by a direct reduction from 3-SAT [11]. In the Computational Complexity Blog, Gasarch [12] asks whether there is a reduction from HAM to its planar version via some planarizing gadget. In a comment to the blog, David Johnson finds this to be an interesting open problem. Using similar arguments as we used for the perfect matching problem, we give a negative answer to Gasarch's question: there is no planarizing gadget for the Hamiltonian cycle problem. Recently we discovered that this observation was made independently and earlier in a post by Burke [2].

2 Preliminaries

Let $G = (V, E)$ be an undirected graph. A *matching* in G is a set $M \subseteq E$, such that no two edges in M have a vertex in common. A matching M is called *perfect* if every vertex occurs as an endpoint of some edge in M. In the decision problem of *perfect matching*, one has to decide whether G has a perfect matching,

$$PM = \{ G \mid G \text{ has a perfect matching} \}.$$

For a weight function $w : E \to \mathbb{N}$ of the graph, the *weight of a matching M* is defined as $w(M) = \sum_{e \in M} w(e)$.

Sequential algorithms to compute maximum matchings use *augmenting path techniques* [15]. They are described in many textbooks, see for example [20, 17]. We mention some simple facts. Let M and M' be matchings in a graph $G = (V, E)$. Consider the subgraph $G' = (V, M \triangle M')$ of G that contains only the edges in the symmetric difference of M and M'. This graph consists of *alternating paths (with respect to M and M')*. That is, the paths have edges alternating from M and M'. Note that some of these paths can be cycles (i.e., start and end vertex being the same). Also, they are simple and pairwise disjoint. If M and M' are *perfect matchings* in G, then $M \triangle M'$ consists of alternating cycles only.

Problems. Let us now define the other matching problems which we consider.

- *Unique perfect matching*: Given a graph G, decide whether G has precisely one perfect matching.
- *Weighted perfect matching*: Given a graph G, a weight function w on the edges and a number W, decide whether there is a perfect matching in G of weight at most W.
- *Exact perfect matching*: Given a graph G where every edge is colored either red or blue, and a number k, decide whether there is a perfect matching in G with exactly k red edges.
- *Weighted exact perfect matching*: Given a graph G, a weight function w on the edges, and a number W, decide whether there is a perfect matching in G of weight exactly W.
- Mod_k *perfect matching*: Given a graph G, decide whether the number of perfect matchings in G is not zero modulo k.

The unique perfect matching problem is in P [9]. For bipartite graphs it is in NC [19, 14], and for planar graphs it is also in NC [30]. It is an open problem whether the unique perfect matching problem is in NC.

The weighted perfect matching problem is in P [25, 31]. If the weights are polynomially bounded, then the problem is in NC for planar graphs [30].

The exact perfect matching problem is a very puzzling problem: it is not even known to be in P (see, e.g., [28, 32]). It is known to be in RNC [27] and in NC for planar graphs [30].

The weighted exact perfect matching problem with polynomially bounded weights is (logspace) equivalent to the exact perfect matching problem. To reduce from the latter to the former we do the following: in a given red-blue graph G, assign weight 1 to each red edge and weight 0 to each blue edge. Then a perfect matching with weight k is a perfect matching with k red edges in G. The reduction in the other direction is also simple: in a given weighted graph G, replace each edge $e = (a, b)$ with a simple path of length $2w(e) - 1$ from a to b. Color the edges of the path with red and blue colors alternatingly, such that there are $w(e)$ red and $w(e) - 1$ blue edges. Only polynomial number of edges are added. A perfect matching with W red edges corresponds to a perfect matching of weight W in G.

In contrast, the weighted exact perfect matching problem in general, i.e., with weights exponential in the number of nodes, is NP-complete. This is mentioned in [28] without a proof. In fact, we can present a reduction from the subset sum problem which shows that the problem becomes hard already with a simple underlying graph structure: the problem is NP-complete for weighted graphs that consist of disjoint copies of 4-cycles. Hence, in this case the general problem reduces to the planar problem. As we show, such a reduction is not possible using planarizing gadgets.

The counting class #P is defined as the class of functions that can be written as $acc_M(x) : \Sigma^* \to \mathbb{N}$, where M is a nondeterministic polynomial time Turing machine and $acc_M(x)$ is the number of accepting computations of M on input x. As shown in [29], it is complete for #P to compute $pm(G)$, the number of perfect matchings of a given bipartite graph G [29]. Counting modulo some integer k leads to the complexity class $\mathrm{Mod}_k\mathrm{P}$ of all problems that can be written as

$$\{\, x \in \Sigma^* \mid acc_M(x) \not\equiv 0 \pmod{k} \}.$$

$\oplus$P is a more common name for $\mathrm{Mod}_2\mathrm{P}$. Over GF(2), the permanent of a matrix is the same as the determinant. That is, Mod_2 perfect matching in bipartite graphs can be computed in NC. Therefore, Mod_2 perfect matching is unlikely to be complete for $\oplus$P. On the other hand, it can be seen that Mod_k perfect matching is complete for $\mathrm{Mod}_k\mathrm{P}$ for every odd $k \geq 3$ (cf. Valiant [29])

Planarizing Gadgets. Let G be a given non-planar graph and consider a drawing of G in the plane. A *planarizing gadget* is a planar graph which is used to replace crossing edges of this drawing of G as shown in Fig. 1. The gadget graph has four designated vertices $v_1, \ldots, v_4$, called *external vertices* which are identified with the corresponding vertices from the crossing. The other vertices of the gadget are called *internal*.

The gadget is independent of the structure of the graph. Hence, every crossing of edges is replaced by a copy of the same gadget. Let G' be the resulting planar graph. The gadget is called *planarizing* for a language L of graphs if

$$G \in L \iff G' \in L. \tag{1}$$

More generally L may be a language of pairs $\langle G, k \rangle$, where G is a (possibly weighted) graph and k is a parameter. Then in the planarizing reduction it is

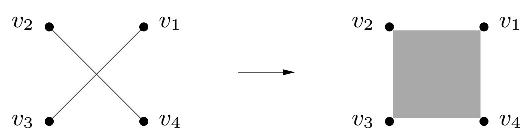


Fig. 1. Planarizing gadget: the two crossing edges on the left are replaced by a planar graph which is indicated by the gray box on the right.

suitable to allow a modification of the parameter k with respect to the number of gadgets introduced by the reduction. We call the (possibly weighted) gadget graph *planarizing* for L if $\langle G, k\rangle \in L \iff \langle G', k'\rangle \in L$, where k' may depend on k, the number t of crossings in the considered drawing of G and the number n of nodes. Also, in case of weighted graphs the weights in the gadget may depend on the weights of the crossing edges and the reduction may modify the weights of G using a linear function depending on t and n.

For our purpose where we want to show that no planarizing gadget exists, it suffices to consider the case when each edge crosses at most one other edge. We will show that even for this case there is no planarizing gadget for various languages L.

3 Perfect Matching Problems

First, we look more closely at the properties of a planarizing gadget for perfect matching problems.

Note that it suffices to consider the case where the gadget contains a single edge connected to v_i, for $i \in [4] = \{1, 2, 3, 4\}$. For if there would be several connections from nodes of the gadget to v_i, we could introduce a new node y_i to the gadget and redirect these edges to y_i instead of v_i. Then we add one more node x_i to the gadget and connect it via the path (v_i, x_i, y_i). Now this modified gadget has the structure from Fig. 2 and there is a direct correspondence between the perfect matchings in both gadgets.

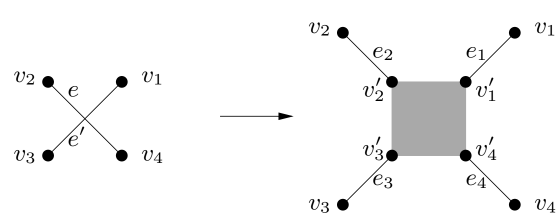


Fig. 2. More details on the planarizing gadget.

As shown in Fig. 2, let $e = (v_2, v_4)$ and $e' = (v_1, v_3)$ be the crossing edges in G and let v_i' be the node in the gadget that is connected with v_i via edge e_i, for $i \in [4]$.

Definition 1. *For $I \subseteq [4]$ let $\mathcal{M}_I$ be the set of matchings M of a gadget that cover all internal nodes of the gadget, and $M \cap \{e_1, e_2, e_3, e_4\} = \{ e_i \mid i \in I\}$.*

The legal matchings *are the matchings that belong to a set in $\mathcal{L}$, where*

$$\mathcal{L} = \{\mathcal{M}_\emptyset, \mathcal{M}_{\{1,3\}}, \mathcal{M}_{\{2,4\}}, \mathcal{M}_{[4]}\}.$$

The illegal matchings *are the matchings that belong to a set in $\mathcal{I}$, where*

$$\mathcal{I} = \{\mathcal{M}_{\{1,2\}}, \mathcal{M}_{\{2,3\}}, \mathcal{M}_{\{3,4\}}, \mathcal{M}_{\{1,4\}}\}.$$

The next Lemma 1 states that in a planarizing gadget for PM legal matchings need to exists and illegal matchings cannot exist which is the actual reason behind naming these classes as legal and illegal. The existence of legal matchings also implies that the gadget needs to have an even number of nodes. This directly implies that $\mathcal{M}_I = \emptyset$ for odd $|I|$ and we do not need to consider these sets.

Lemma 1. *A gadget is planarizing for PM only if*

- *each set in $\mathcal{L}$ is non-empty, and*
- *there is no illegal matching.*

Proof. Consider Fig. 3. Parts (a), (b), and (c) show that $\mathcal{M}_{[4]}$, $\mathcal{M}_{\{2,4\}}$, and $\mathcal{M}_\emptyset$

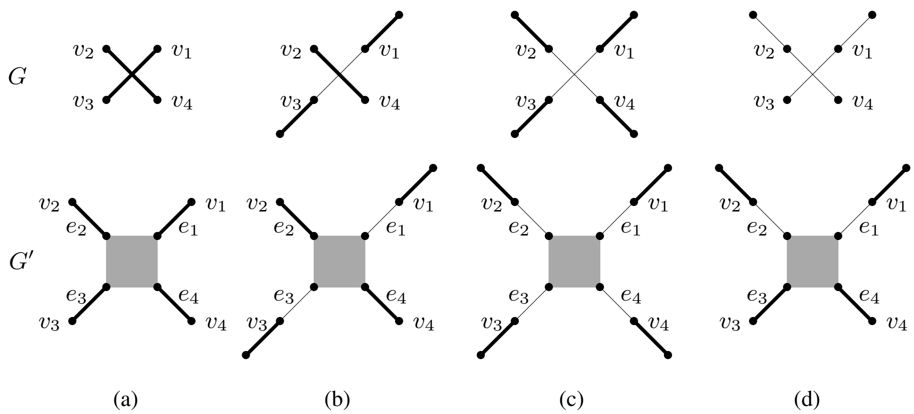


Fig. 3. Graphs G with a perfect matching that contains (a) both, (b) one, and (c) none of the crossing edges. Matching edges are drawn with bold lines. Note that, for graphs G' to have a perfect matching, the gadget should have the legal matchings which contain (a) all four, (b) two opposite, and (c) none of the edges $e_1, \ldots, e_4$. In (d), graph G has no perfect matching. If the gadget would allow the illegal perfect matching that contains e_3, e_4 and not e_1, e_2, then the resulting graph G' would have a perfect matching. Hence, such a gadget does not work.

should be non-empty (respectively). The case $\mathcal{M}_{\{1,3\}}$ is symmetric to $\mathcal{M}_{\{2,4\}}$.

Part (d) shows that a gadget which allows an illegal matching (a matching in $\mathcal{M}_{\{3,4\}}$) is not planarizing for *PM*. The cases where two other neighboring edges of $e_1, \dots, e_4$ are used, are symmetric. Therefore, no illegal matching is allowed to exist. □

In the proof of Lemma 1, we argued with the graphs shown in Fig. 3. For simplicity, these graphs are planar, but are drawn with two edges crossing. Clearly, the gadget has to work also in such cases, and hence, we do not need to deal with more complicated non-planar graphs. However, it is easy to extend our graphs to non-planar graphs in such a way, that the perfect matchings are preserved: Let G be one of the above graphs. For every pair of non-adjacent nodes u, v in G, we add two additional nodes $x_{u,v}, y_{u,v}$ which are connected by an edge, and connect u and v with $y_{u,v}$. Let G^* be the resulting graph. Since the only neighbor of $x_{u,v}$ is $y_{u,v}$, every perfect matching in G^* has to use edge $(x_{u,v}, y_{u,v})$. The other edges in the perfect matching are all from G. Hence, perfect matchings in G and G^* differ only by the newly introduced edges $(x_{u,v}, y_{u,v})$.

If G has n nodes, then G^* has the complete graph K_n as minor. Therefore, G^* is non-planar for $n \geq 5$. Only the graph in Fig. 3 (a) has just 4 nodes. But it is easy to enlarge it by a few extra nodes and still cover the same case. Hence, things do not change if we restrict our arguments to non-planar graphs only.

3.1 Perfect Matching

Next, we show that no planarizing gadget for the perfect matching problem exists. The proof constructs an illegal perfect matching out of legal ones.

Theorem 1. *There is no planarizing gadget for the perfect matching problem.*

Proof. Suppose there is a planarizing gadget. We refer to the denotation in Fig. 2 and Definition 1. According to Lemma 1 there are legal matchings $M_{1,3} \in \mathcal{M}_{\{1,3\}}$ and $M_{2,4} \in \mathcal{M}_{\{2,4\}}$.

Consider the subgraph with edges $M_{1,3} \triangle M_{2,4}$ of the gadget: as explained in the preliminary section, $M_{1,3} \triangle M_{2,4}$ consists of some alternating cycles and paths. The nodes v_1, v_2, v_3, v_4 must lie on alternating paths. Since the two matchings cover all nodes in the gadget, there are precisely two disjoint alternating paths p and q, each of which connects two nodes in $\{v_1, v_2, v_3, v_4\}$. The remaining edges of $M_{1,3} \triangle M_{2,4}$ form alternating cycles.

Let p denote the path that contains node v_1. We distinguish three cases:

(i) Suppose that p connects v_1 with v_3. Therefore, q connects v_2 with v_4. As we assume that there is a planar drawing of the gadget where v_1, v_2, v_3, v_4 are placed like in Fig. 2, the two paths must cross in at least one common vertex. Since p and q are disjoint, this is not possible.

(ii) Suppose that $p = p_{1,2}$ connects v_1 with v_2, and $q = p_{3,4}$ connects v_3 with v_4. From $M_{1,3}$ and $M_{2,4}$ we now construct two illegal matchings $M_{2,3}$ and $M_{1,4}$ by exchanging the edges on path $p_{1,2}$ between these two sets. Let $E(p_{1,2})$ denote the set of edges on path $p_{1,2}$. We define

$$M_{2,3} = M_{1,3} \triangle E(p_{1,2}).$$

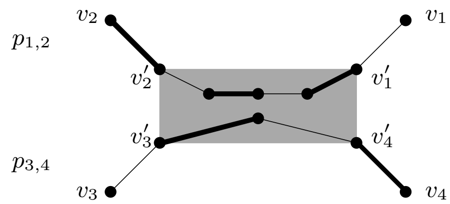


Fig. 4. Matchings $M_{1,3}$ and $M_{2,4}$ are indicated, $M_{2,4}$ with bold edges. The upper alternating path $p = p_{1,2}$ connects v_1 with v_2, the lower path $q = p_{3,4}$ connects v_3 with v_4. The illegal matching $M_{2,3}$ is defined as the bold edges on $p_{1,2}$ and the non-bold edges on $p_{3,4}$ and the other edges from $M_{1,3}$ that are not on these paths. $M_{1,4}$ consists of the remaining edges on both paths and the other edges from $M_{2,4}$.

Similarly we define $M_{1,4} = M_{2,4} \triangle E(p_{1,2})$. Fig. 4 gives an example of the construction. Now both matchings $M_{1,3}$ and $M_{2,4}$ cover each internal node of the gadget, and

- $e_2, e_3 \in M_{2,3}$ and $e_1, e_4 \notin M_{2,3}$ and
- $e_1, e_4 \in M_{1,4}$ and $e_2, e_3 \notin M_{1,4}$.

Hence, $M_{2,3}$ and $M_{1,4}$ are illegal. Therefore, this case is not possible either.

(iii) The case that p connects v_1 with v_4 is analogous to case (ii).

Hence, all cases lead to a contradiction. Therefore, no such gadget exists. □

3.2 Unique Perfect Matching

A planarizing gadget for the unique perfect matching problem needs to have the property that in each of the four legal cases, the matching inside the gadget must be unique, i.e., each set in $\mathcal{L}$ of Definition 1 contains exactly one element. Otherwise, it would not maintain uniqueness in Fig. 3 (a)–(c). However, as shown in the proof of Theorem 1, we cannot avoid getting additional illegal matchings in the gadget. This can be used to destroy the uniqueness in G'. The details can be found in the full version of the paper.

Corollary 1. *There is no planarizing gadget for the unique perfect matching problem.*

3.3 Weighted Perfect Matching

A planarizing gadget for weighted perfect matching may have illegal matchings. But it can be seen that in each class $\mathcal{M} \in \mathcal{L}$ of legal matchings there is a matching $M \in \mathcal{M}$ whose weight is smaller than the weight of each illegal matching. The proof of Theorem 1 can be extended to show that this is not possible. The details can be found in the full version of the paper.

Corollary 2. *There is no planarizing gadget for the weighted perfect matching problem.*

3.4 Exact Perfect Matching

Corollary 2 says that no planarizing gadget can preserve the minimum weight perfect matching. But it might still be possible that a gadget can preserve some exact weight, which is neither minimum nor maximum.

When replacing crossings of equal weight edges, it can be seen that there are matchings $M_{1,3} \in \mathcal{M}_{\{1,3\}}$ and $M_{2,4} \in \mathcal{M}_{\{2,4\}}$ with some fixed weights and all illegal matchings in the gadget have different weights. The proof of Theorem 1 can be extended to show that the gadget has two illegal matchings $M_{2,3}$, $M_{1,4}$ such that $w(M_{2,3}) + w(M_{1,4}) = w(M_{1,3}) + w(M_{2,4})$.

If a graph G is drawn with $t \geq 2$ crossings, then we will have t gadgets in G'. When the reduction increases the weight by some W_t then there is a combination with two illegal matchings that gives the same increasing weight. We can use this to show that there is no planarizing gadget that works correctly for all graphs. A detailed proof can be found in the full version of the paper.

Theorem 2. *There is no planarizing gadget for the weighted exact perfect matching problem.*

The proofs of Corollary 2 and Theorem 2 already show the non-existence of a planarizing gadget for the case when all edge weights are equal which corresponds to the perfect matching problem. Hence, we can formulate the following corollary.

Corollary 3. *There is no planarizing gadget that reduces the perfect matching problem to the planar weighted perfect matching problem or the planar weighted exact perfect matching problem.*

Similarly the exact perfect matching problem is a special case of the exact weighted perfect matching problem.

Corollary 4. *There is no planarizing gadget for the exact perfect matching problem. Moreover, there is no planarizing gadget that reduces the exact perfect matching problem to the planar weighted exact perfect matching problem.*

3.5 Mod$_k$ Perfect Matching

In the preliminary section we already mentioned that Mod$_k$ perfect matching (for short, Mod$_k$-*PM*), is complete for Mod$_k$P for odd $k \geq 3$. Hence, there is no planarizing gadget for Mod$_k$-*PM* nor any other NC computable planarizing reduction, unless Mod$_k$P = NC, for odd $k \geq 3$. We prove the non-existence of a planarizing gadget independent of the Mod$_k$P $\neq$ NC assumption, for $k \geq 3$.

For a gadget to reduce a graph G to its planarized version G', we must have $pm(G) \equiv 0 \pmod{k}$ if and only if $pm(G') \equiv 0 \pmod{k}$. From the graphs in Fig. 3 it follows that we must have

- $|\mathcal{M}| \not\equiv 0 \pmod{k}$ for all $\mathcal{M} \in \mathcal{L}$ and
- $|\mathcal{M}| \equiv 0 \pmod{k}$ for all $\mathcal{M} \in \mathcal{I}$.

A planarizing gadget for Mod_2-*PM* has been provided by [5]. In the following lemma we observe that the legal types of matching classes all have the same size modulo k, say a, and a is relatively prime to k. The proof is omitted here.

Lemma 2. *For a planarizing gadget for* Mod_k*-PM there is a number* a *such that* $|\mathcal{M}| \equiv a \pmod{k}$ *for all* $\mathcal{M} \in \mathcal{L}$*. Moreover,* $\gcd(a, k) = 1$.

Our next goal is to construct a bijection between pairs of legal and illegal matchings of a gadget. Recall the proof of Theorem 1: we started with two legal matchings $M_0 \in \mathcal{M}_{1,3}$ and $M_1 \in \mathcal{M}_{2,4}$. Then we defined p to be the alternating path in $M_0 \triangle M_1$ that contains node v_1, and matchings $M_2 = M_0 \triangle E(p)$ and $M_3 = M_1 \triangle E(p)$. Path p either ends in v_2 or in v_4.

- If p ends in v_2 then $M_2 \in \mathcal{M}_{2,3}$ and $M_3 \in \mathcal{M}_{1,4}$,
- if p ends in v_4 then $M_2 \in \mathcal{M}_{3,4}$ and $M_3 \in \mathcal{M}_{1,2}$.

Now observe that this process is reversible: we have $M_2 \triangle M_3 = M_0 \triangle M_1$. That is, $M_2 \triangle M_3$ defines the same alternating path p through v_1 and $M_2 \triangle E(p) = M_0$ and $M_3 \triangle E(p) = M_1$.

The same argument will work if we start with legal matchings $M_0 \in \mathcal{M}_\emptyset$ and $M_1 \in \mathcal{M}_{[4]}$. Hence, we constructed a bijection between the following sets:

$$\mathcal{S} = (\mathcal{M}_\emptyset \times \mathcal{M}_{[4]}) \;\cup\; (\mathcal{M}_{\{1,3\}} \times \mathcal{M}_{\{2,4\}})$$
$$\mathcal{T} = (\mathcal{M}_{\{1,2\}} \times \mathcal{M}_{\{3,4\}}) \;\cup\; (\mathcal{M}_{\{1,4\}} \times \mathcal{M}_{\{2,3\}})$$

We conclude:

Lemma 3. *For a planarizing gadget we have* $|\mathcal{S}| = |\mathcal{T}|$.

Theorem 3. *There is no planarizing gadget for* Mod_k*-PM for* $k \geq 3$.

Proof. By Lemma 2, we have $|\mathcal{S}| \equiv 2a^2 \pmod{k}$. Since $\mathcal{T}$ contains only illegal classes of matchings, we have $|\mathcal{T}| \equiv 0 \pmod{k}$. By Lemma 3, it follows that $2a^2 \equiv 0 \pmod{k}$. But since $\gcd(a, k) = 1$, this is not possible for $k \geq 3$. □

4 Hamiltonian Cycle

A *Hamiltonian cycle* in graph G is a simple cycle that visits every node in G. The Hamiltonian cycle problem, HAM, is to decide whether a given graph G has a Hamiltonian cycle. A proof can be found in the full version of the paper.

Theorem 4. *There is no planarizing gadget for the Hamiltonian cycle problem.*

In a straightforward way one can modify the proof of Theorem 4 to obtain similar results for the (directed) Hamiltonian path problem and the directed Hamiltonian cycle problem.

Corollary 5. *There is no planarizing gadget for the directed Hamiltonian cycle problem nor for the (directed) Hamiltonian path problem.*

A similar argument shows that there is no planarizing gadget for reachability.

5 Discussion

Our approach allowed us to show unconditionally that there are no planarizing gadgets for various graph problems. Clearly, this does not imply that there is no logspace reduction from the general problem to its planar version. For example, for the Hamiltonian cycle problem or the exact weighted perfect matching problem with large weights, the general and the planar versions are both NP-complete. Nonetheless, we think that the observations are interesting and give some new insight into the problems. Moreover, for the problems like perfect matching where it is not clear whether the problem reduces to its planar version, we eliminated some plausible approach to a reduction.

In our approach, we assumed that the planarizing gadget should work for basically every drawing of the input graph in the plane. A major improvement would be to show that there is no (logspace computable) drawing of the input graph for which a planarizing gadget exists. In fact such a statement can be made for k-colorability with $k \geq 4$: there is no planarizing gadget for the 5-clique K_5, irrespective of the drawing of K_5. Such a gadget would guaranty that the planarized version K'_5 is non-4-colorable, which is not possible. On the other hand, such an unconditional statement does not hold for the perfect matching problem nor for the Hamiltonian cycle problem. For these problems there are drawings that allow a planarizing gadget: If one is able to compute a Hamiltonian cycle while computing a drawing of a graph, one can draw the graph such that all edges that belong to the Hamiltonian cycle do not cross any other edge (start by drawing the cycle as circle). Similarly, for the perfect matching problem there is a drawing where matching edges do not have crossings. For such drawings the empty graph is a planarizing gadget (just remove the crossing edges). The following question arises: if one assumes that the Hamiltonian cycle, resp. the perfect matching, of a graph G cannot be computed in logspace, can one show that there is no planarizing gadget for any logspace computable drawing of G?

Acknowledgment We thank an anonymous referee for his helpful comments.

References

[1] M. Agrawal. On derandomizing tests for certain polynomial identities. In *Proceedings of the Conference on Computational Complexity*, pages 355–359, 2003.

[2] K. Burke, 2012. `http://cstheory.stackexchange.com/questions/9587`.

[3] E. Dahlhaus, P. Hajnal, and M. Karpinski. On the parallel complexity of Hamiltonian cycles and matching problem in dense graphs. *J. Algorithms*, 15:367–384, 1993.

[4] E. Dahlhaus and M. Karpinski. Matching and multidimensional matching in chordal and strongly chordal graphs. *Disc. Appl. Math.*, 84:79–91, 1998.

[5] S. Datta, R. Kulkarni, N. Limaye, and M. Mahajan. Planarity, determinants, permanents, and (unique) matchings. *ACM Trans. Comput. Theory*, 1:10:1–10:20, 2010.

[6] S. Datta, R. Kulkarni, and S. Roy. Deterministically isolating a perfect matching in bipartite planar graphs. *Theor. Comput. Syst.*, 47:737–757, 2010.

[7] S. Datta, R. Kulkarni, and R. Tewari. Perfect matching in bipartite planar graphs is in UL. Technical Report TR10-201, ECCC, 2011.

[8] J. Edmonds. Paths, trees, and flowers. *Can. J. Math.*, 17:449–467, 1965.

[9] H. N. Gabow, H. Kaplan, and R. E. Tarjan. Unique maximum matching algorithms. *J. Algorithms*, 40(2):159 – 183, 2001.

[10] M. R. Garey, D. S. Johnson, and L. Stockmeyer. Some simplified NP-complete graph problems. *Theor. Comput. Sci.*, 1(3):237–267, 1976.

[11] M. R. Garey, D. S. Johnson, and R. E. Tarjan. The planar Hamiltonian circuit problem is NP-complete. *SIAM J. Comput*, 5(4):704–714, 1976.

[12] W. Gasarch. Is there a nice gadget for showing planar HC is NPC? Computational Complexity Blog, `http://blog.computationalcomplexity.org/2012/01/is-there-nice-gadget-for-showing-planar.html`, 2012.

[13] T. M. Hoang. On the matching problem for special graph classes. In *Proceedings of the Conference on Computational Complexity*, pages 139–150, 2010.

[14] T.M. Hoang, M. Mahajan, and T. Thierauf. On the bipartite unique perfect matching problem. In *Proceding of the International Colloquium on Automata, Languages and Programming*, LNCS 4051, pages 453–464. Springer, 2006.

[15] J. Hopcroft and R. Karp. An $n^{5/2}$ algorithm for maximum matchings in bipartite graphs. *SIAM J. Comput.*, 2:225–231, 1973.

[16] R. M. Karp. Reducibility among combinatorial problems. In *Complex. Computer Comput.*, pages 85–103. Plenum Press, 1972.

[17] M. Karpinski and W. Rytter. *Fast Parallel Algorithms for Graph Matching Problems*. Oxford University Press, 1998.

[18] P. W. Kasteleyn. Graph theory and crystal physics. In F. Harary, editor, *Graph Theory and Theoret. Physics*, pages 43–110. Academic Press, 1967.

[19] D. C. Kozen, U. V. Vazirani, and V. V. Vazirani. NC algorithms for comparability graphs, interval graphs, and testing for unique perfect matchings. In *Proceedings of 5th FSTTCS*, pages 496–503. Springer, 1985.

[20] Dexter Kozen. *The Design and Analysis of Algorithms*. Springer, 1991.

[21] R. Kulkarni, M. Mahajan, and K. Varadarajan. Some perfect matchings and perfect half-integral matchings in NC. *Chic. J. Theor. Comput.*, 2008(4), 2008.

[22] G. Lev, M. Pippenger, and L. Valiant. A fast parallel algorithm for routing in permutation networks. *IEEE Trans. Computers*, 30:93–100, 1981.

[23] L. Lovasz and M. D. Plummer. *Matching theory*. North-Holland, 1986.

[24] M. Mahajan and K. R. Varadarajan. A new NC-algorithm for finding a perfect matching in bipartite planar and small genus graphs. In *32th ACM Symp. Theo. Comput. (STOC)*, pages 351–357. ACM Press, 2000.

[25] S. Micali and V. Vazirani. An $O(\sqrt{|v|} \cdot |E|)$ algorithm for finding maximum matching in general graphs. In *FOCS*, pages 17–27, 1980.

[26] G. L. Miller and J. S. Naor. Flow in planar graphs with multiple sources and sinks. *SIAM J. Comput.*, 24:1002–1017, 1995.

[27] K. Mulmuley, U. Vazirani, and V. Vazirani. Matching is as easy as matrix inversion. *Combinatorica*, 7:105–113, 1987.

[28] C. H. Papadimitriou and M. Yannakakis. The complexity of restricted spanning tree problems. *J. ACM*, 29:285–309, 1982.

[29] L. Valiant. The complexity of computing the permanent. *Theoretical Computer Science*, 8:189–201, 1979.

[30] V. Vazirani. NC algorithms for computing the number of perfect matchings in $K_{3,3}$-free graphs and related problems. *Inf. Comput.*, 80:152–164, 1989.

[31] V. Vazirani. A theory of alternating paths and blossoms for proving correctness of the $O(\sqrt{V}E)$ general graph maximum matching algorithm. *Combinatorica*, 14:71–109, 1994.

[32] R. Yuster. Almost exact matchings. *Algorithmica*, 63(1-2):39–50, 2012.

A Appendix

A.1 NP-completeness of Weighted Exact Perfect Matching

Fig. 5 presents a reduction from the subset sum problem to the weighted exact perfect matching problem which shows that the problem becomes hard with a trivial underlying graph structure.

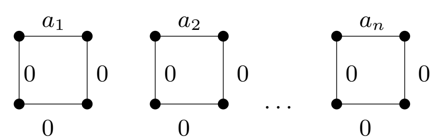


Fig. 5. Reduction from subset sum to exact perfect matching. Given an instance $a_1, a_2, \ldots, a_n, b$ of integers for the subset sum problem, we construct the weighted graph G shown above, which consists of n copies of C_4 with weights as indicated. The subset sum instance has a solution, i.e., there is an $S \subseteq \{1, \ldots, n\}$ with $\sum_{i \in S} a_i = b$, if and only if G has a perfect matching of weight b.

The subset sum problem is known to be NP-complete even under logspace reductions, like all the NP-complete problems in [16]. Clearly, also the reduction in Fig. 5 is computable in logspace, and the constructed graph is bipartite and planar. Therefore, the bipartite planar weighted exact perfect matching problem is NP-complete under logspace reducibility.

A.2 Proof of Corollary 2

Proof. To show the non-existence it suffices to consider the case when all the edge weights in the given graph are same, say 1. In this case, any two perfect matchings in the reduced graph G', will have any difference in the weights only inside the gadget. Hence, we do not need to consider the weights outside the gadget.

Now, it suffices to reconsider case (ii) in the proof of Theorem 1. We have two legal matchings $M_{1,3}, M_{2,4}$ and two illegal matchings $M_{2,3}$, $M_{1,4}$. The illegal matchings are allowed to exist, but the weight of each illegal matching has to be strictly larger than the weight of all legal matchings.

Since $M_{2,3}$ and $M_{1,4}$ are constructed from $M_{1,3}, M_{2,4}$ solely by exchanging some edges between the two sets, i.e., $M_{2,3} \cap M_{1,4} = M_{1,3} \cap M_{2,4}$ and $M_{2,3} \triangle M_{1,4} = M_{1,3} \triangle M_{2,4}$, we have

$$w(M_{2,3}) + w(M_{1,4}) = w(M_{1,3}) + w(M_{2,4}).$$

But this contradicts our assumption that the weight of $M_{2,3}$, $M_{1,4}$ is strictly larger than that of $M_{1,3}, M_{2,4}$. $\square$

A.3 Proof of Theorem 2

Proof. As argued in the proof of Corollary 2, we do not need to consider the weights of the graph outside the gadget. Let G be a graph that is drawn with $t \geq 2$ crossings. Then graph G' contains t gadgets. We pick two of the gadgets in G'. It suffices to reconsider case (ii) in the proof of Theorem 1 for both gadgets.

- Let $M_{1,3}, M_{2,4}$ be two legal matchings and $e_1, \ldots, e_4$ be the connecting edges in the first gadget, and let $M_{2,3}$, $M_{1,4}$ be two illegal matchings as constructed in the proof of Theorem 1.
- Let $M'_{1,3}, M'_{2,4}, M'_{2,3}, M'_{1,4}$ and $e'_1, \ldots, e'_4$ denote corresponding matchings and connecting edges in the other gadget.

Define $\overset{\circ}{M}_{i,j} = M_{i,j} \setminus \{e_i, e_j\}$ to be matching $M_{i,j}$ without the connecting edges e_i, e_j. The legal matchings all have weight W_0 inside the gadget. For example $w(\overset{\circ}{M}_{1,3}) = W_0$. The illegal matchings should have weights different from W_0 inside the gadget.

We have the following equations for the weights of the illegal matchings:

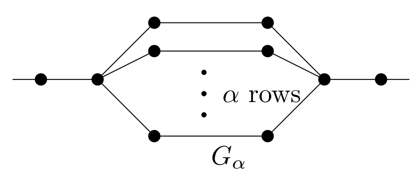
1. $w(\overset{\circ}{M}_{2,3}) = w(\overset{\circ}{M'}_{2,3})$ and $w(\overset{\circ}{M}_{1,4}) = w(\overset{\circ}{M'}_{1,4})$,
2. $w(\overset{\circ}{M}_{2,3}) + w(\overset{\circ}{M}_{1,4}) = w(\overset{\circ}{M}_{1,3}) + w(\overset{\circ}{M}_{2,4}) = 2W_0$.

Define matching $M = M_{2,3} \cup M'_{1,4}$. Then M covers both gadgets, is illegal, and M has weight $2W_0$ inside the two gadgets. Now we can extend M by legal matchings of weight W_0 each in the other gadgets in G'. This gives an illegal matching of weight tW_0 in all the gadgets together. But for the reduction to work, all legal matchings should have weight different from all illegal matchings. □

A.4 Proof of Lemma 2

Proof. We first construct a *conditional multiplier gadget* G_α for $\alpha \geq 1$.

The multiplier G_α on the right has two external edges and is intended to replace an edge in the original graph, such that the number of matchings that use this edge is multiplied by α and the number of matchings that don't use this edge stays the same.



To see that $|\mathcal{M}_{[4]}| \equiv |\mathcal{M}_{\{2,4\}}| \pmod{k}$, consider the graph $G_{\alpha,\gamma}$ in Fig. 6. It has $pm(G_{\alpha,\gamma}) = \alpha + \gamma$ matchings.

After replacing the crossing by the planarizing gadget, the number of matchings of the planarized graph $G'_{\alpha,\gamma}$ is

$$pm(G'_{\alpha,\gamma}) = \alpha\,|\mathcal{M}_{\{2,4\}}| + \gamma\,|\mathcal{M}_{[4]}|\,.$$

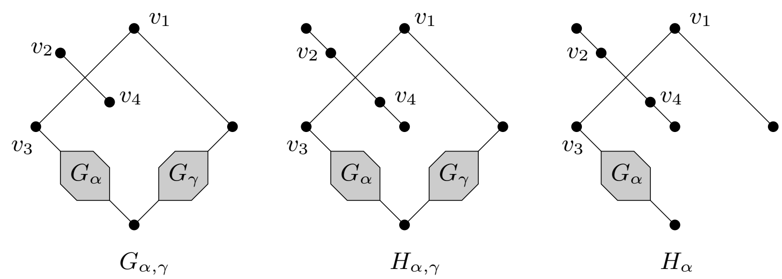


Fig. 6. The graphs shown contain the multiplier gadget from above, indicated by the shaded boxes. Graph $G_{\alpha,\gamma}$ has $\alpha + \gamma$ perfect matchings. γ of them use both crossing edges, (v_1, v_3) and (v_2, v_4), and α of them just contain (v_2, v_4).
Graph $H_{\alpha,\gamma}$ has $\alpha + \gamma$ perfect matchings as well. α of them use none of the crossing edges and γ of them just contain (v_1, v_3).
Graph H_α has α perfect matchings and none of them contains the crossing edges.

We choose $\alpha = |\mathcal{M}_{[4]}| \mod k$ and $\gamma = k - (|\mathcal{M}_{\{2,4\}}| \mod k)$. Then $pm(G'_{\alpha,\gamma}) \equiv 0 \pmod k$ and therefore, because it is a reduction, $pm(G_{\alpha,\gamma}) \equiv 0 \pmod k$. Hence, we get $|\mathcal{M}_{\{2,4\}}| \equiv |\mathcal{M}_{[4]}| \pmod k$.

By symmetry we get that $|\mathcal{M}_{\{1,3\}}| \equiv |\mathcal{M}_{[4]}| \pmod k$. By a similar argument using graph $H_{\alpha,\gamma}$ shown in Fig. 6, we get $|\mathcal{M}_{\emptyset}| \equiv |\mathcal{M}_{\{1,3\}}| \pmod k$. Therefore, modulo k, all the legal classes have the same size a as claimed.

Now let $d = \gcd(a, k)$. Consider the Graph H_α in Fig. 6. Let $\alpha = k/d$. For the planarized version H'_α we have $pm(H'_\alpha) = \alpha\, a = \frac{a}{d}\, k \equiv 0 \pmod k$. Because we have a reduction, this implies $pm(H_\alpha) = k/d \equiv 0 \pmod k$. Therefore, $d = 1$. □

A.5 Proof of Theorem 4

Proof. Assume that there is a planarizing gadget. Consider a drawing of the 8-cycle C_8 that contains one crossing, see Fig. 7. Since C_8 has a Hamiltonian cycle,

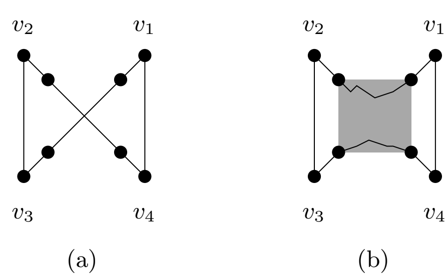


Fig. 7. (a) A drawing of C_8 with one crossing and (b) its planarization

also the planarized version of the drawing must have a Hamiltonian cycle C. Since C is simple, it cannot have a crossing in the gadget. Thus cycle C must look like the graph in Fig. 7 (b). As a consequence, the gadget must contain two paths p_{12} and p_{34} that connect v_1 with v_2 and v_3 with v_4, respectively, such that (i) p_{12} and p_{34} are disjoint and (ii) p_{12} and p_{34} together cover each node in the gadget.

By symmetry (consider the drawing rotated by 90°) there exists a similar pair of paths p_{23} and p_{14}.

Now consider the drawing of the graph G in Fig. 8 (a) which consists of two copies of C_7 . After replacing the crossings with the planarizing gadget, the

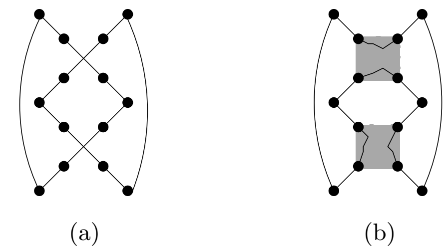


Fig. 8. (a) A drawing of a graph consisting of two copies of C_7 and (b) its planarization

graph would have a Hamiltonian cycle, whereas G does not. □

For simplicity we argued with planar graphs in the proof of Theorem 4. However, it is easy to transform our graphs to non-planar graphs in such a way, that the Hamiltonian cycle is preserved. For this, consider a graph $G = (V, E)$ and replace each original node $w \in V$ by a path w_1, w_2, w_3, w_4, w_5, and each edge $\{w, w'\} \in E$ by the edges $\{w'_5, w_1\}$ and $\{w_5, w'_1\}$. Clearly, there is a one to one correspondence for the Hamiltonian cycles. Then add edges $\{w_3, w'_3\}$ for all $w, w' \in V$ which yields a clique of size $|V|$ in the transformed graph. Notice that no Hamiltonian cycle in the transformed graph can traverse through an edge $\{w_3, w'_3\}$, since then one of the nodes in $\{w_2, w'_2, w_4, w'_4\}$ would remain unvisited.

A.6 Figures

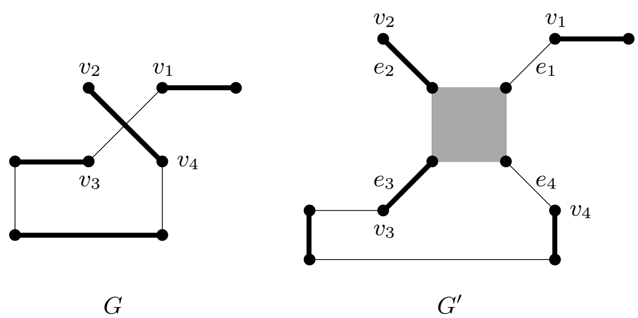


Fig. 9. Graph G has a unique perfect matching. As shown in the proof of Theorem 1, the gadget will have an illegal matching M with $e_2, e_3 \in M$ and $e_1, e_4 \notin M$. The matching M can be extended to a perfect matching in the resulting graph G'. M is an additional perfect matching beside the originally unique perfect matching. The uniqueness is lost.

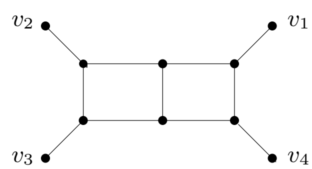


Fig. 10. Planarizing gadget for Mod$_2$-*PM* provided by [5]. Here we have $|\mathcal{M}| \equiv 1 \pmod 2$ for every $\mathcal{M} \in \mathcal{L}$ and $|\mathcal{M}| \equiv 0 \pmod 2$ for every $\mathcal{M} \in \mathcal{I}$.