A pre-Weekend Talk on Online Learning

TGIF Talk Series

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Outline

- Some Motivating Examples
  - Discovering customer preferences
  - Learning investment profiles
  - Detecting credit card fraud

- The Formal Online Learning Framework
  - Notion of regret
  - Formalization of motivating examples

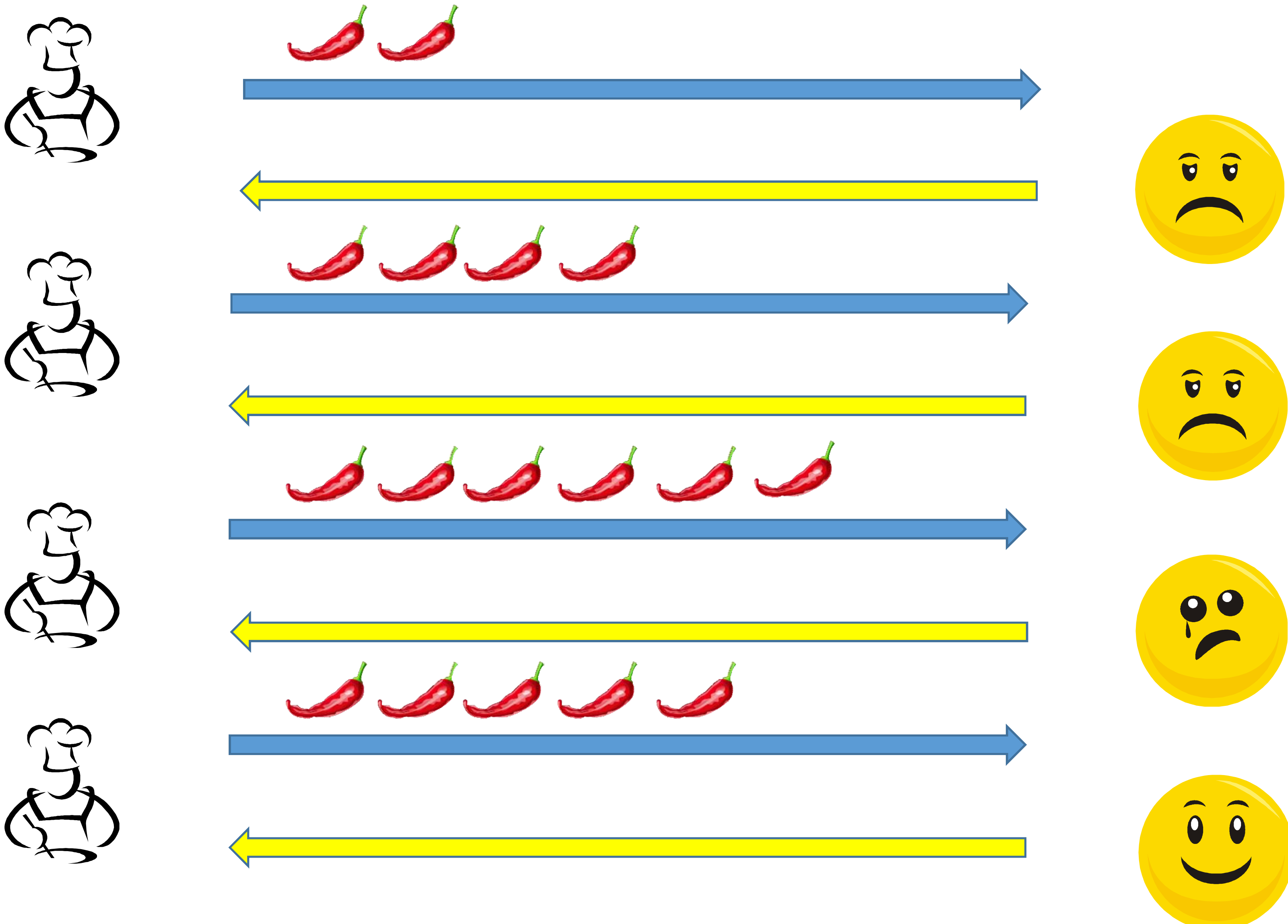
- Simple Online Algorithms
  - Online classification, regression
  - Online ranking
  - Batch solvers for large scale learning problems

- Other “Feedback-based” Learning Frameworks

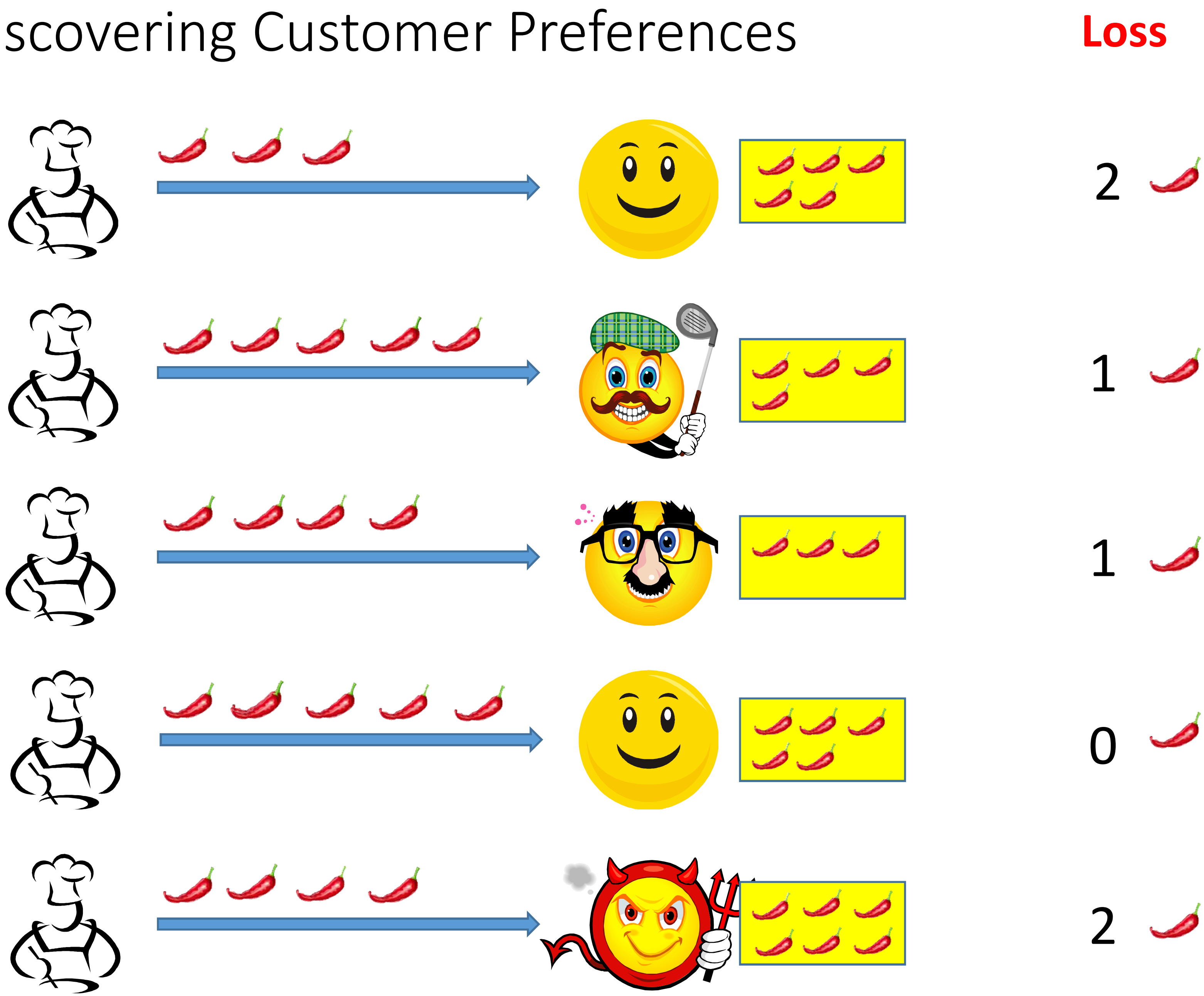
Some Motivating Examples

Why Online Learning can be Useful

The Cook's Dilemma

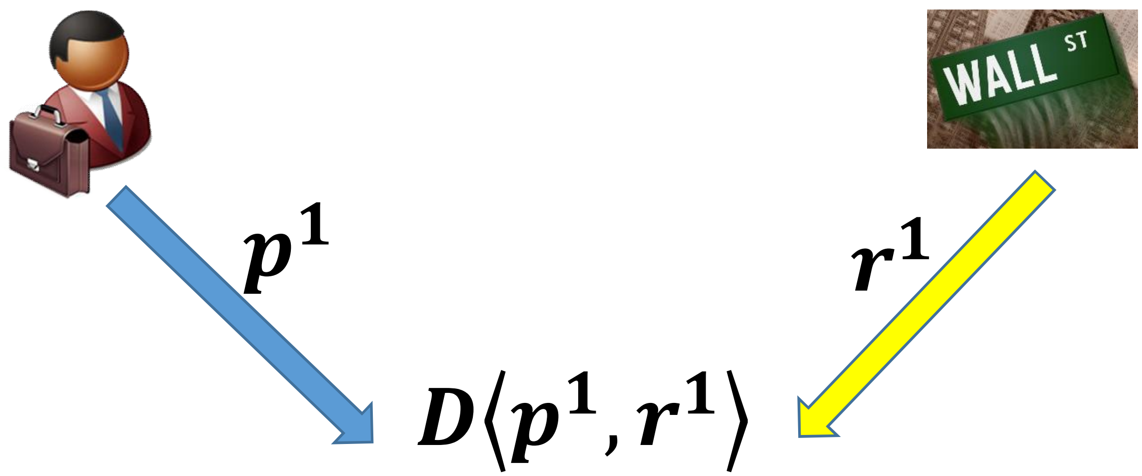


Discovering Customer Preferences



Learning Investment Profiles

- k assets $a_1, a_2, \dots, a_k$ that give returns proportional to investment
- Asset a_i gives back $\$r_i$ as return per dollar invested
  - If I invest $\$d_i$ in a_i then total return is $\sum d_i r_i = d^\top r$
  - Return profile r depends on market forces, other investors and keeps changing
- I have corpus of $\$D$ that I decide to invest completely in these assets
  - Let p_i decide proportion of investment in asset a_i, i.e. investment is $p_i D$



- Corpus at time T becomes $D \prod_{t=1}^{T} \langle p^t, r^t \rangle$: reward to be maximized

Detecting Credit Card Fraud

- Classify credit card payments into $\{+,-\}$
  - Each payment $\mathrm{p} \in \mathcal{P}$ is described by a vector $x_p \in \Re^d$
  - Other problems such as branch prediction/churn prediction
- Linear classification model
  - Choose $w \in \Re^d$ and classify p as $\operatorname{sign}(w^\top x_p)$
- Online process; at each time t
  - A credit card payment p_t is detected
  - We propose a linear classifier w_t and classify p_t as $\operatorname{sign}(w_t^\top x_{p_t})$
  - True status of payment y_t is made known and our mistake (if any) is revealed
- Wish to minimize the number of mistakes made by us
  - Wish to propose a "good" sequence of w

The Formal Online Learning Framework

How we assess Online Learning Algorithms

The Online Learning Model

- An attempt to model an interactive and adaptive environment
  - We have a set of actions $\mathcal{A}$
  - Environment has a set of loss functions $\mathcal{L} = \{\ell: A \to \Re_+\}$

- In each round t
  - We play some action $a_t \in \mathcal{A}$
  - Environment responds with a loss function $\ell_t \in \mathcal{L}$
  - We are forced to incur a loss $\ell_t(a_t)$
  - Environment can adapt to our actions (or even be adversarial)

- Our goal: minimize cumulative loss $\sum_{t=1}^{T} \ell_t(a_t)$
  - Can cumulative loss be brought down to zero : mostly no !
  - More reasonable measure of performance: single best action in hindsight
  - Regret: $R_T \coloneqq \sum_{t=1}^{T} \ell_t(a_t) \; - \min_{a \in \mathcal{A}} \sum_{t=1}^{T} \ell_t(a)$
  - Why is this a suitable notion of performance ?

Motivating Examples Revisited

- Detecting customer preferences
  - Assume we can represent customer $c \in \mathcal{C}$ as a vector $x_c \in \Re^d$
  - Set of actions are linear functions predicting spice levels for that customer
$$\hat{s}_c = w^\top x_c$$
  - Loss function given by squared difference between true and preferred spiciness
$$\ell_{\text{abs}}(w, x_c) = (\hat{s}_c - s_c)^2$$
  - At time step t customer c_t comes and $\ell_t(w_t) = \ell_{\text{abs}}(w_t, x_{c_t})$
  - Goal: make customers as happy as the single best spice level
- Credit card fraud detection
  - Actions are the set of linear classifiers $\mathcal{W} = \{w \in \Re^d\}$
  - Loss functions are mistake functions
$$\ell_{0/1}(w, x_p) = \mathbb{I}\{y_p w^\top x_p < 0\} = \begin{cases} 1 & \text{if } y_p \neq \text{sign}(w^\top x_p) \\ 0 & \text{otherwise} \end{cases}$$
$$\ell_t(w_t) = \ell_{0/1}(w_t, x_{p_t})$$
  - Detection of credit card fraud might change buying profiles (adversarial)
  - Goal: make (almost) as few mistakes as single best classifier

Motivating Examples Revisited

- Learning investment profiles
  - Set of actions is the d-dimensional simplex $\mathcal{A} = \{p \in \Re^d, p \geq 0, \|p\|_1 = 1\}$
  - Reward received at t^{th} step is $\langle p^t, r^t \rangle$ where r^t is the return given by market
  - Total reward (assume w.l.o.g. initial corpus is $D = 1$)

$$\prod_{t=1}^{T} \langle p_t, r_t \rangle = \exp\left(\sum_{t=1}^{T} \log \langle p_t, r_t \rangle \right)$$

  - Returns affected by investment, other market factors (adaptive, adversarial)
  - Can think of $\ell(p, r) = -\log\langle p, r \rangle$ as a negative reward or a loss

$$\ell_t(p_t) = -\log\langle p_t, r_t \rangle$$

  - Regret (equivalently) given by

$$\mathcal{R}_T = \sum_{t=1}^{T} \ell(p_t, r_t) - \min_{p \in \mathcal{A}} \sum_{t=1}^{T} \ell(p, r_t)$$

  - Goal: make as much profit as the single best investment profile in hindsight

Simple Online Algorithms

What makes online learning click ?

Online Linear Classification

- Perceptron Algorithm

1. Start with $w_0 = 0$
2. Classify o_t as $\operatorname{sign}\left(w_{t-1}^\top x_{o_t}\right)$
3. If correct classification i.e. $y_t = \operatorname{sign}\left(w_t^\top x_{o_t}\right)$, then let $w_t = w_{t-1}$
4. Else $w_t = w_{t-1} + y_t x_{o_t}$

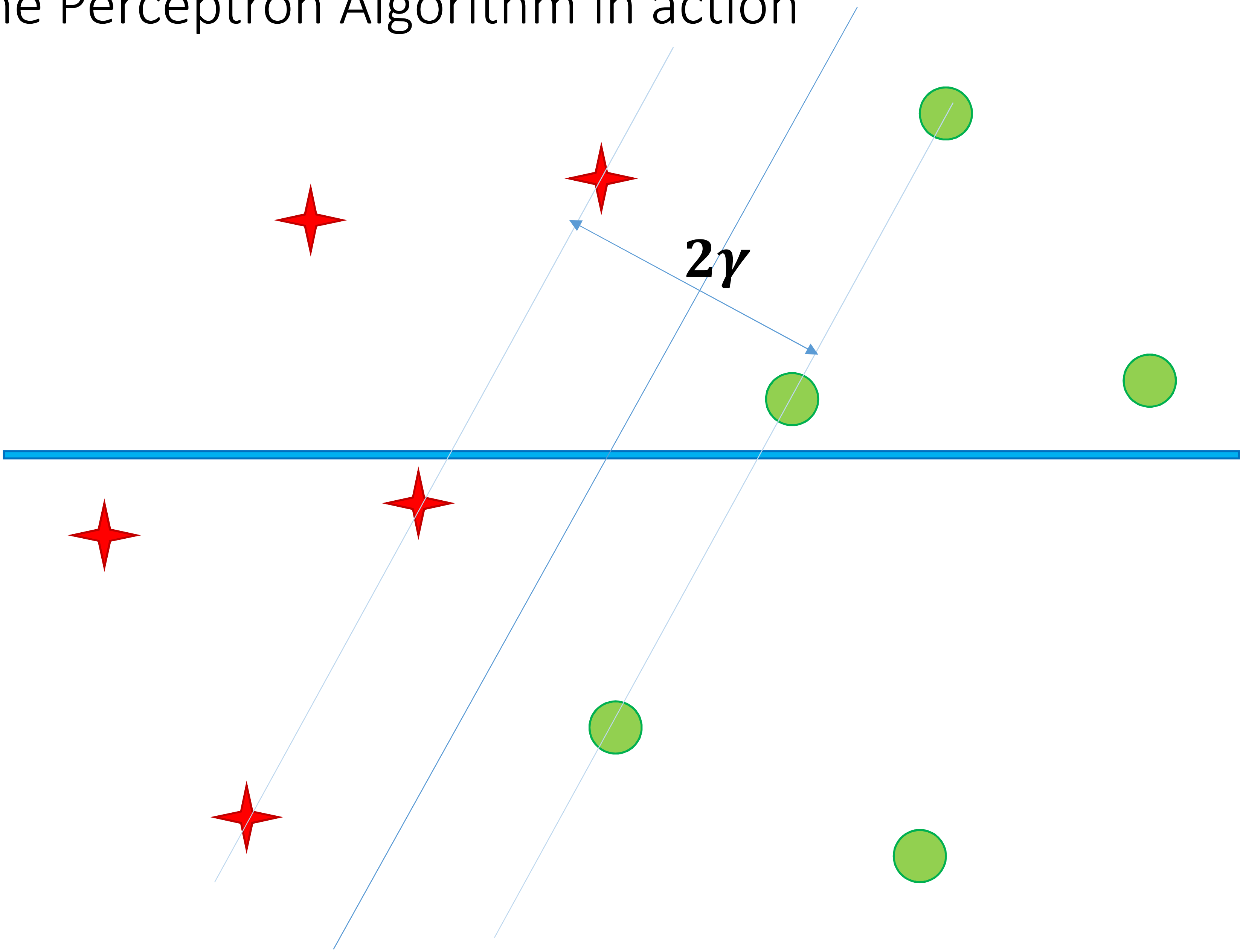
- Loss function $\ell_{0/1}(w, o) = \mathbb{I}\{y_o w^\top x_o < 0\}$ i.e. 1 iff w misclassifies o
- If there exists a perfect linear separator w^* such that $\mathrm{y_t w^{*\top} x_{o_t}} \geq \gamma$,

$$\mathcal{R}_T = \sum \ell_{0/1}(w_t, o_t) - \sum \ell_{0/1}(w^*, o_t) \leq \frac{1}{\gamma^2}$$

- If there exists an imperfect separator w^* such that $y_t {w^*}^\top x_{o_t} \geq \gamma - \xi_t$,

$$\mathcal{R}_T = \sum \ell_{0/1}(w_t, o_t) - \sum \ell_{0/1}(w^*, o_t) \leq \frac{1}{\gamma^2} + \frac{1}{\gamma} \sum \xi_t$$

The Perceptron Algorithm in action



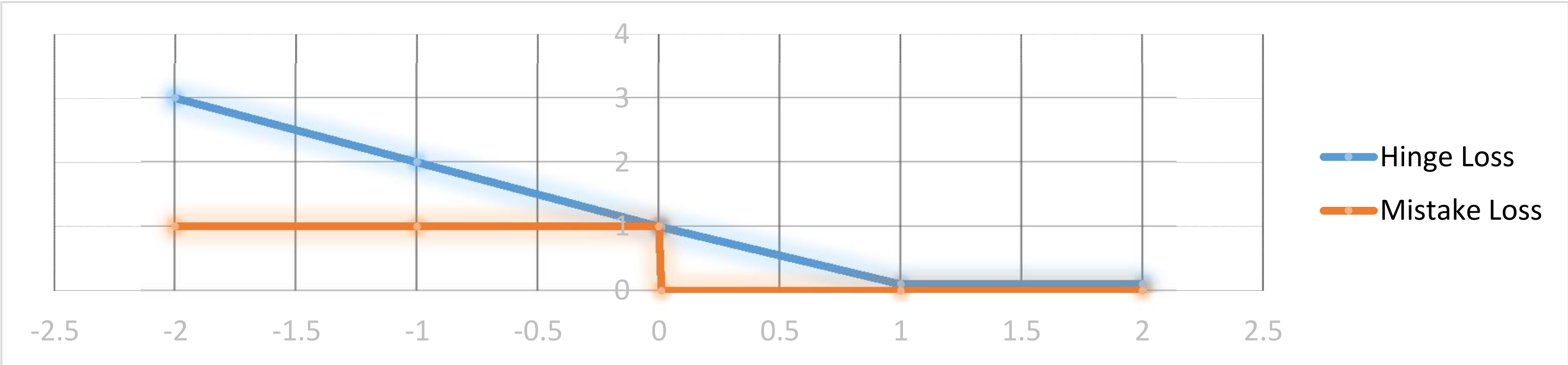
Online Regression

- The Perceptron Algorithm was (almost) a gradient descent algorithm
- Consider the loss function

$$\ell_{\text{hinge}}(w, x) = \max\{1 - yw^\top x, 0\}$$

- $\tilde{\ell}$ is a convex *surrogate* to the mistake function $\ell_{0/1}(w, x) = \mathbb{I}\{yw^\top x < 0\}$

$$\ell_{\text{hinge}}(w, x) \geq \ell_{0/1}(w, x)$$



- When perceptron makes a mistake i.e. $\ell_{0/1}(w, x) = 1$, we have

$$\nabla_w \ell_{hinge}(w, x) = -yx$$

- Thus the perceptron update step $w_t = w_{t-1} + y_t x_{o_t}$ is a gradient step !

Online Regression via Online Gradient Descent

- Suppose we are taking actions $a_t \in \mathcal{A}$ and receiving losses $\ell_t \in \mathcal{L}$
  - Assume that all loss function $\ell_t : \mathcal{A} \to \Re_+$ are convex and Lipchitz
  - Examples $\ell_t(a) = (a^\top x_t - y_t)^2$, $\ell_t(a) = -\log(a^\top x_t)$, $\ell_t(a) = [1 - y_t a^\top x_t]_+$

- Online Gradient Descent (for linear predictions problems)

1. Start with $a_0 = 0$
2. Receive object x_t and predict value $a_{t-1}^\top x_t$ for object x_t
3. Receive loss function ℓ_t and update $a_t = a_{t-1} - \frac{1}{\sqrt{t}} \nabla_a \ell_t(a_{t-1})$
   - Some more work needed to ensure that $a_t \in \mathcal{A}$ as well

- We can ensure that

$$\mathrm{R_T} = \sum_{t=1}^{T} \ell_t(a_t) - \min_{a \in \mathcal{A}} \sum_{t=1}^{T} \ell_t(a) \leq \mathcal{O}\left(\sqrt{T}\right)$$

Online Bipartite Ranking

- Documents $d_1, d_2, \dots d_t, \dots$ arrive in a continuous stream to be ranked
- Each document in labelled either "relevant" (+) or "irrelevant" (-)
- Goal: somehow rank all relevant documents before irrelevant ones
- Method: assign relevance score $r: d_t \to r_t$ to document d_t and sort

- We incur loss for "swaps" $\ell_{\text{rank}}(r, d_t, d_{t'}) = \mathbb{I}\{(y_t - y_{t'})(r_t - r_{t'}) < 0\}$
  - Minimize number of swaps $\sum_{t=1}^{T} \sum_{t'=1}^{T} \ell_{\text{rank}}(d_t, d_{t'})$
  - Problem is equivalent to maximizing area under the ROC curve of TP/FP

- Challenges
  - No reference point: no "true" relevance score
  - Need pairs of documents to learn a scoring function: get only singles
  - Solution: keep (some) of the past points in a buffer to construct pairs on the fly
  - Several interesting algorithmic and theoretical problems still open

Batch Solvers

- Statistical learning gets a batch of randomly chosen training examples
$$(x_1, y_1), \dots, (x_n, y_n) \sim \mathcal{X} \times \mathcal{Y}$$
- We wish to learn a function $f \in \mathcal{F}$ that does well on these examples
$$\min_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^{n} \ell(f, x_i)$$
where $\ell : \mathcal{F} \times \mathcal{X} \to \Re_+$ is a loss function (classification, regression etc)
- Statistical Learning Theory: such an f does well on unseen points as well!
- Solving "batch" problem may be infeasible: $n \gg 1$, distributed storage etc.
- Solution: solve the online problem instead
- E.g. online gradient descent will solve for a $f_i \in \mathcal{F}$ such that
$$\sum_{i=1}^{n} \ell(f_i, x_i) \leq \min_{f \in \mathcal{F}} \sum_{i=1}^{n} \ell(f, x_i) + \mathcal{R}_n$$
where $\mathcal{R}_n = o(n)$

Batch Solvers

- Thus we have an $f_t \in \mathcal{F}$ such that

$$\frac{1}{\mathrm{n}}\sum_{i=1}^{n}\ell(f_i, x_i) \leq \min_{f\in\mathcal{F}}\frac{1}{n}\sum_{i=1}^{n}\ell(f, x_i) + \epsilon$$

  where $\epsilon = \frac{\mathcal{R}_n}{n} = o(1)$
- Online to batch conversion bounds
  - Argue for the performance of $\hat{f} = \frac{1}{n}\sum_{i=1}^{n} f_i$ on random unseen points
- Expected loss of $\hat{f}$ on a random unseen point is bounded

$$\mathbb{E}_x[\![\ell(\hat{f}, x)]\!] \leq \frac{\mathcal{R}_n}{n} + \mathcal{O}\left(\frac{1}{\sqrt{n}}\right)$$

- Several batch solvers e.g. PEGASOS, MIDAS, LASVM use techniques such as Stochastic online gradient descent for large scale learning

Other Feedback based Learning Frameworks

- Two axes of variation: modelling of environment and feedback
- Online Learning: some modelling of environment and full feedback
  - Losses are simple functions over linear models (can be made non linear though)
  - At each step the loss function itself is given to us: full information
  - Models are agnostic: no realizability assumptions are made

- Multi-armed Bandits: weak modelling of environment, weak feedback
  - Often no assumptions made on nature of loss function
  - Limited feedback: only loss value on played action made available
  - Contextual bandits try to model loss function but make realizability assumptions

- Reinforcement Learning: Strong modelling of environment, weak feedback
  - Environment modelled as a state space with adaptive stochastic transitions
  - Reward functions modeled as functions of state space and action
  - Limited feedback available: need to learn, state space as well as reward function