Random Projection Trees Revisited

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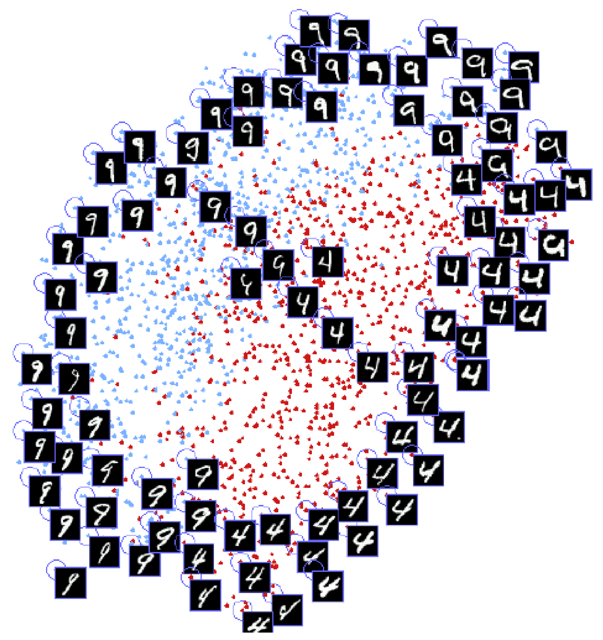
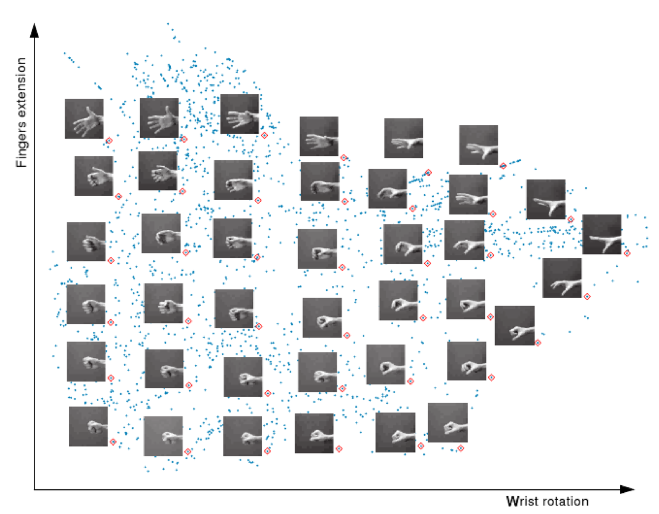
Introduction

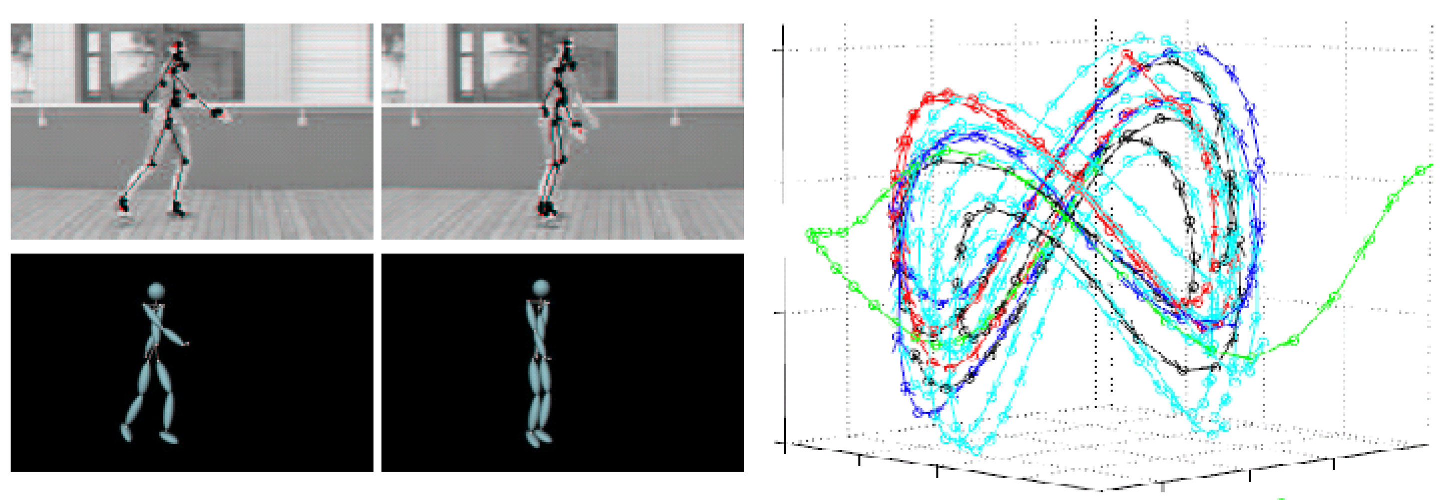
- Goal : Exploit geometric structure in data
- Manifold structure seen in several applications



(a) Digit Recognition[a]



(b) Gesture Identification[b]



(c) Motion Capture[c]

[a] Hadsell *et al*, Dimensionality Reduction by Learning an Invariant Mapping, CVPR 2006.
[b] Tenenbaum *et al*, A Global Geometric Framework for Nonlinear Dimensionality Reduction, Science 290(5500) : 2319-2323, 2000.
[c] Urtasun *et al*, 3D People Tracking with Gaussian Process Dynamical Models, CVPR 2006.

Random Projection Tree[d]

- A space partitioning data structure that adapts to manifolds
- Simple partitioning plan – uses random hyperplanes
- Assumes no information about the manifold – fully adaptive
- Guaranteed reduction in cell size after a certain number of levels

Theorem 1.

Given a cell C of RPTREE-MAX of radius Δ containing data of doubling dimension d, with probability at least $\frac{1}{2}$, every cell $\mathcal{O}(d \log d)$ levels below has radius $\frac{\Delta}{2}$ or less.

- A d-dimensional Riemannian manifold has $\mathcal{O}(d)$ doubling dimension - hence RPTREE-MAX adapts to manifolds
- Applications to regression, spectral clustering, face recognition

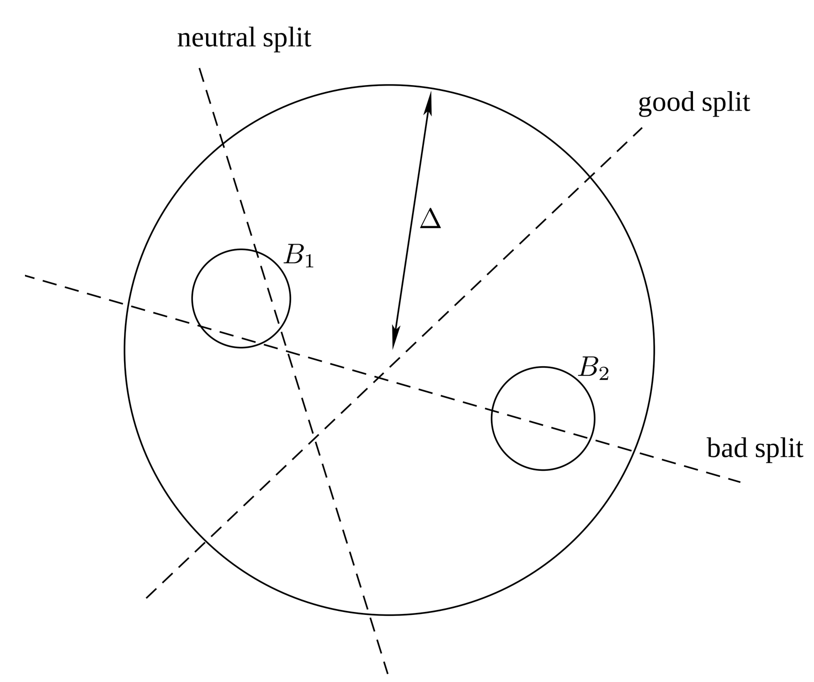
[d] Dasgupta and Freund, Random Projection Trees and Low dimensional Manifolds, STOC 2008.

Our Results for the RPTREE-MAX

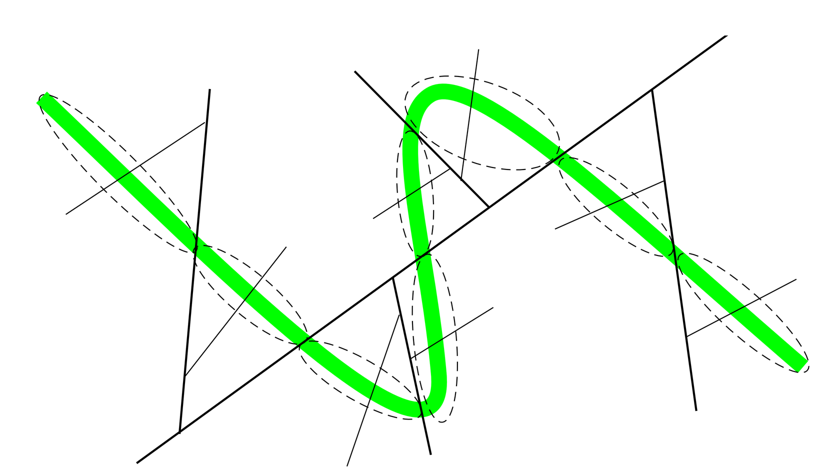
Theorem 2. (Space Partitioning)

For any $s \geq 2$, given a cell C of RPTREE-MAX of radius Δ containing data of doubling dimension d, with probability $\frac{1}{2}$, every cell $\mathcal{O}(d \log s \log sd)$ levels below has radius $\frac{\Delta}{s}$ or less.

- Cover the manifold with small balls of size $\frac{\Delta}{s\sqrt{d}}$
- $\mathcal{O}\left((sd)^d\right)$ balls suffice due to bounded doubling dimension
- Random hyperplanes split every well-separated pair of balls



(d) B_1 and B_2 have radius $\Delta/s\sqrt{d}$ and their centers are $> \Delta/s$ apart



(e) RPTREE acting on a manifold

Theorem 3. (Packing Bound)

Given a fixed ball B of radius R, with probability greater than $\frac{1}{2}$, the number of disjoint RPTREE-MAX cells of radius greater than r that intersect B is at most $\left(\frac{R}{r}\right)^{\mathcal{O}(d \log d \log(dR/r))}$.

- w.h.p. B is contained in a cell of size $\mathcal{O}\left(Rd\sqrt{d}\log d\right)$
- Use Theorem 2 to bound the number of children of this cell of size r
- Effectively this is an aspect ratio bound of $\mathcal{O}\left(d\sqrt{d}\log d\right)$

Comparing the Guarantees

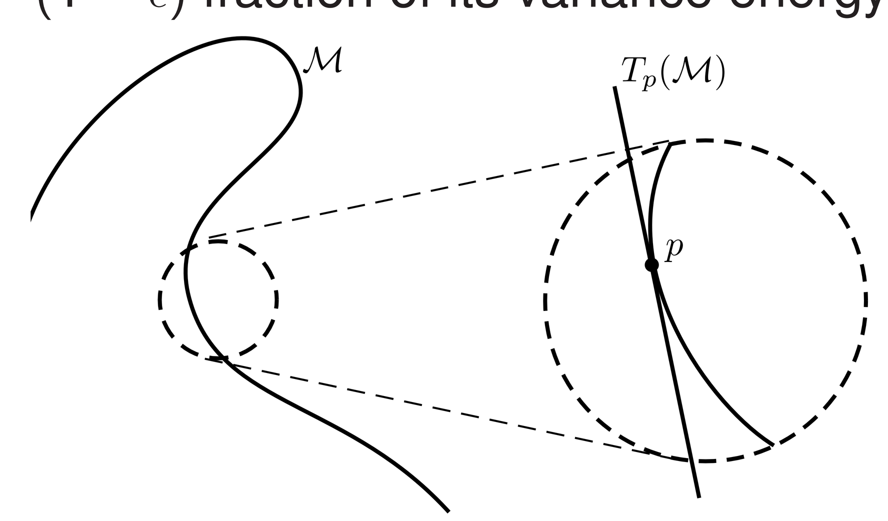
	(k-d) BBD Tree	RPTREE-MAX
$L(s)$	$D \log s$	$\tilde{\mathcal{O}}\left(d \log^2 s\right)$
$P(R, r)$	$\left\lceil 1 + \frac{6R}{r} \right\rceil^D$	$\left(\frac{R}{r}\right)^{\tilde{\mathcal{O}}(d)}$

d : manifold dimensionality, D : ambient dimensionality
The $\tilde{\mathcal{O}}$ () notation hides poly-log factors in d and R/r

- RPTREE-MAX guarantees do not depend on ambient dimensionality
- The guarantees hold with constant confidence that can be boosted
- Thus RPTREE-MAX is space partitioning as well as has packing bounds

Our Results for the RPTREE-MEAN

- The RPTREE-MEAN adapts to local covariance dimension (LCD)
- A set has LCD (d, ϵ, r) if when restricted to any ball of radius r, a $(1 - \epsilon)$ fraction of its variance energy lies in some d dimensions.



- Most of the energy for manifolds lies in the tangent plane
- Hence RPTREE-MEAN adapts to manifolds as well

Theorem 4. (Local Covariance Dimension of Manifolds)

For any $\epsilon \leq \frac{1}{4}$, a d-dimensional Riemannian manifold $\mathcal{M}$ with condition number τ has local covariance dimension $\left(d, \epsilon, \frac{\sqrt{\epsilon\tau}}{3}\right)$.

Space Partitioning Data Structures

- Lots of them around : k-d tree, BBD tree, BAR tree
- Useful in approximate NN searches and clustering
- Typical guarantees given :
  - **Space Partitioning** : After $L(s)$ levels, sizes of cells go down by a factor of s
  - **Bounded Aspect Ratio** : Cells in the tree have bounded aspect ratio
  - **Packing Bound** : Given a fixed ball B of radius R, there exists a bound $P(R, r)$ on number of disjoint cells of size r that intersect B
  - **Bounded Depth** : A tree constructed using n data points has $\mathcal{O}(\log n)$ depth

Future Work

- Applications require space partitioning data structures that have packing bounds as well as bounded depth
- RPTREE-MAX can be coerced into becoming left/right-deep by an adversarial placement of points
- Can one modify RPTREE-MAX so that it becomes space partitioning, has packing bounds as well as bounded depth