CS 682: Quantum Computing	Lecture 2: July 31, 2008
Lecturer: Piyush P. Kurur	*Scribe: Lovy Singhal*

2.1 Hilbert spaces

Before we begin with some serious mathematics, we briefly digress to share an anecdote concerning the subject matter of our today's study. *Hilbert Spaces* arose as a result of the pioneering idea given by David Hilbert to consider 'square summable infinite sequences' as "points" in an n-dimensional space, while he was working with integral equations. Some time later, Hilbert, quite notorious for his absent-mindedness, attended a conference with Richard Courant at which several of the papers presented kept on referring to Hilbert spaces. After one such presentation, Hilbert turned to Courant and asked a curious question: "Richard, exactly what is a Hilbert Space?" [1].

In the discussion that follows, we assume familiarity with elementary linear algebra. In particular, the notions of linear mappings, vector spaces, orthonormal basis *et cetera* are taken to be known to all.

Definition 2.1.1. *A* ***Hilbert space*** $\mathcal{H}$ *is defined to be a vector space over the field* $\mathbb{C}$ *of complex numbers with an inner product* $\langle\ ,\ \rangle : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}$ *defined on it such that for all* $\underline{u}, \underline{v}, \underline{w} \in \mathbb{C}$

1. $\langle\, \underline{u}\, ,\, \underline{v}\, \rangle = \overline{\langle\, \underline{v}\, ,\, \underline{u}\, \rangle}$
2. $\langle\, \underline{u}\, ,\, \alpha\, \underline{v} + \beta\, \underline{w}\, \rangle = \alpha \langle\, \underline{u}\, ,\, \underline{v}\, \rangle + \beta\, \langle\, \underline{u}\, ,\, \underline{w}\, \rangle$
3. $\langle\, \alpha\, \underline{u} + \beta\, \underline{v}\, ,\, \underline{w}\, \rangle = \bar{\alpha}\, \langle\, \underline{u}\, ,\, \underline{w}\, \rangle + \bar{\beta}\, \langle\, \underline{v}\, ,\, \underline{w}\, \rangle$
4. $\langle\, \underline{u}\, ,\, \underline{u}\, \rangle \in \mathbb{R}$, $\langle\, \underline{u}\, ,\, \underline{u}\, \rangle \geq 0$ *and* $\langle\, \underline{u}\, ,\, \underline{u}\, \rangle = 0$ *iff* $\underline{u} = \underline{0}$

Example 2.1. Consider the n-dimensional vector space $(\mathbb{C}^n, +)$ over $\mathbb{C}$ with the vector addition defined as the usual component-wise addition of vectors. The standard inner product on this vector space is defined as

$$\langle (x_1, x_2, \ldots, x_n)\, , (y_1, y_2, \ldots, y_n) \rangle = \sum_{i=1}^{n} \bar{x}_i y_i. \tag{2.1}$$

Together with this inner product and the vector addition $+$, $\mathbb{C}^n$ is a Hilbert space.

In general, Hilbert spaces may be finite or infinite dimensional. But for our endeavour, it is sufficient to confine ourselves to finite dimensional Hilbert spaces though we shall occasionally pause in between to make a remark or two about infinite dimensional Hilbert spaces.

Example 2.2. An example of an infinite dimensional Hilbert space is the set of all complex-valued functions over the real interval $[0, 1]$

$$\{\, f \mid f : [0,1] \rightarrow \mathbb{C} \,\}$$

with vector addition and inner product defined as follows

$$\forall\, x \in [0,1] \;\; (f+g)\,(x) = f(x) + g(x)$$
$$\langle f, g \rangle = \int_0^1 \bar{f} g \,\mathrm{dx} \tag{2.2}$$

Definition 2.1.2. *For any linear map $A : \mathcal{H} \rightarrow \mathcal{H}$, the* ***adjoint*** *of $A = A^*$ is given by $\langle\, A^*\underline{u} \,,\, \underline{v} \,\rangle = \langle\, \underline{u} \,,\, A\underline{v} \,\rangle$.*

The following remarks about Hilbert spaces are in order:

Remark 2.1.1. For an infinite dimensional vector space over $\mathbb{C}$ to be a Hilbert space, it should possess the property of completeness, i.e., every sequence of vectors in $\mathcal{H}$ which is Cauchy should converge to some limit in $\mathcal{H}$.

Remark 2.1.2. The second and third conditions stated in Definition 2.1.1 render the inner product $\langle\;,\;\rangle$ to be a *sesquilinear operator*, which means conjugate linear in the first operand and linear in the second operand.

Remark 2.1.3. If the Hilbert space $\mathcal{H}$ is finite-dimensional, then any linear operator $A : \mathcal{H} \rightarrow \mathcal{H}$ can be represented by an $n \times n$ matrix with entries in $\mathbb{C}$ after fixing a basis of $\mathcal{H}$, and hence the adjoint of $A = A^* = \bar{A}^T$ is well defined. This is a basis-dependent way of defining the adjoint of an operator, though it can be defined in a more general way. The set of all linear operators on a finite-dimensional Hilbert space $\mathcal{H}$ together with the addition operation is itself a finite-dimensional Hilbert space.

Remark 2.1.4. If however, the Hilbert space $\mathcal{H}$ is infinite dimensional, then the adjoint of $A = A^*$ is defined iff A is a continuous linear operator on $\mathcal{H}$. In this case, the well-definedness of A^* is guaranteed by the *Riesz representation theorem.* The set of all linear operators on an infinite dimensional Hilbert space together with the usual addition operation do not form a Hilbert space but the set of all continuous linear operators on an infinite dimensional Hilbert space does form a *Banach space.*

2.1.1 Notations

Physicists usually denote the adjoint A^* of an operator A by the seemingly violent notation $A^\dagger$, and the conjugate $\bar{\alpha}$ of a acalar α by α^*. For all subsequent discussions, we will take the middle path of denoting the adjoint of an operator A by $A^\dagger$ and the conjugate of a scalar α by $\bar{\alpha}$.

2.2 Dirac notation

We now arrive at what is arguably one of the greatest contibutions of physics to humanity – the Dirac notation.

Let $\mathcal{H}$ be a finite-dimensional Hilbert space. The elements of $\mathcal{H}$ are called *ket - vectors* denoted as $|\psi\rangle$ (pronounced as *ket* ψ)

Definition 2.2.1. *A* ***linear form*** *or* ***functional*** *on $\mathcal{H}$ is a linear map from $\mathcal{H}$ to $\mathbb{C}$,*

$$f : \mathcal{H} \to \mathbb{C}$$

$$f\left(\alpha\, \underline{u} + \beta\, \underline{v}\right) = \alpha\, f\left(\underline{u}\right) + \beta\, f\left(\underline{v}\right) \;\; \forall\; \alpha, \beta \in \mathbb{C} \quad \underline{u}, \underline{v} \in \mathcal{H}$$

The space of all linear forms on $\mathcal{H}$ together with the usual addition of maps as the vector addition forms a Hilbert space, known as the *dual vector space* of $\mathcal{H}$, denoted as $\mathcal{H}^*$.

Corresponding to every element $|\psi\rangle \in \mathcal{H}$, there exists a unique $\langle\psi| \in \mathcal{H}^*$, called *bra vector* such that for all $|x\rangle, |y\rangle \in \mathcal{H}$

1. $\langle\psi|\left(|x\rangle\right) = \langle\, \psi\,,\, x\,\rangle \equiv \langle\psi|x\rangle$
2. $|\psi\rangle\langle x| \;\; |y\rangle = \langle x|y\rangle \;\; |\psi\rangle$

Example 2.3. 1. Let us fix an orthonormal basis for an n-dimensional Hilbert space $\mathcal{H}$ as $\{|1\rangle, |2\rangle, \ldots, |n\rangle\}$. We want an operator that on being given $|k\rangle$ as input, produces $|j\rangle$ as output. It requires a little inspection to figure out that $|j\rangle\langle k|$ is one such operator.

2. Dual of $\alpha\,|\psi\rangle$ is $\bar{\alpha}\,\langle\psi|$.

3. Dual of $A\,|\psi\rangle$ is $\langle\psi|\, A^\dagger$.

Remark 2.2.1. $(V^*)^* \cong V$ iff V is a finite-dimensional vector space.

Remark 2.2.2. We had remarked in 2.1.3 that the adjoint of an operator can be defined in more general terms. We further remark that if A is a linear map from $\mathcal{H}$ to $\mathcal{K}$, then $A^\dagger$ can be thought of as a linear map from $\mathcal{H}^*$ to $\mathcal{K}^*$.

References

[1] `http://www.anecdotage.com/index.php?aid=14087`