Bayesian Inference for Gaussians, Working With Gaussians

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Topics in Probabilistic Modeling and Inference (CS698X)

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Recap: Bayesian Inference for Mean of a Gaussian

- Consider N i.i.d. observations $\mathbf{X} = \{x_1, \ldots, x_N\}$ drawn from a one-dim Gaussian $\mathcal{N}(x|\mu, \sigma^2)$

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- **Exercise:** Compute the posterior if $p(\mu) = \mathcal{N}(\mu|\mu_0, \frac{\sigma^2}{\kappa_0})$. Also, what does κ_0 mean intuitively?

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- Exercise: Work out (or look up) the posterior predictive $p(x_*|\mathbf{X})$ in these cases (isn't Gaussian)

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(note: μ and λ are coupled in the Gaussian part)

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- NG is conjugate to Gaussian when both mean & precision are unknown

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An Aside: generalized-t and Student-t distribution

- Obtained if we integrate out the precision of a Gaussian using a conjugate gamma prior

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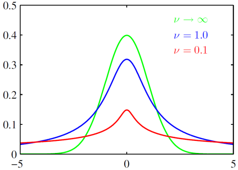

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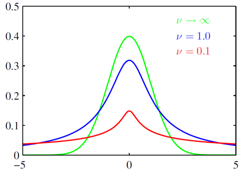

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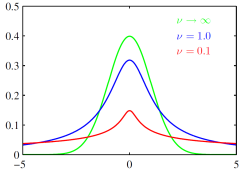

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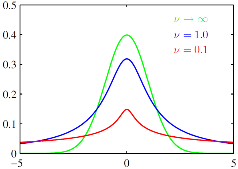

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  - For $\nu \to \infty$, it is equivalent to a Gaussian

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- For further details (e.g., full equations, posterior predictive, etc), refer to "Conjugate Bayesian analysis of the Gaussian distribution" by Murphy (2007), or MLAPP Chapter 4

Some Useful Properties of Gaussians

Multivariate Gaussian: Some Alternative Representations

- The (multivariate) Gaussian with mean $\boldsymbol{\mu}$ and cov. matrix $\boldsymbol{\Sigma}$

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- Note that there is a term quadratic in $\boldsymbol{x}$ (involves $\boldsymbol{\Lambda}=\boldsymbol{\Sigma}^{-1}$) and linear in $\boldsymbol{x}$ (involves $\boldsymbol{\xi}=\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}$)

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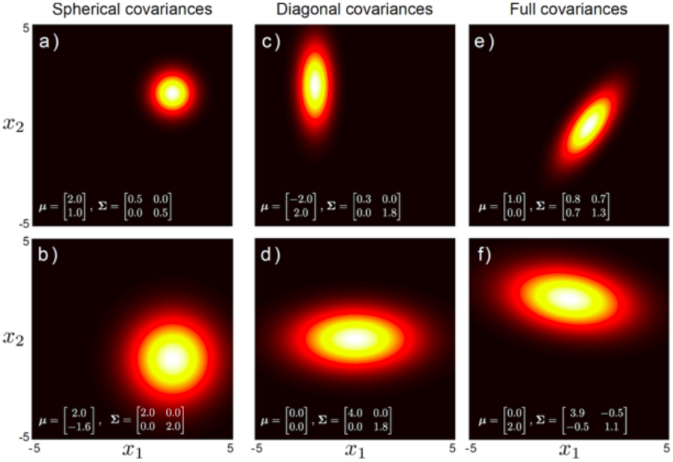
- Note that there is a term quadratic in $\boldsymbol{x}$ (involves $\boldsymbol{\Lambda} = \boldsymbol{\Sigma}^{-1}$) and linear in $\boldsymbol{x}$ (involves $\boldsymbol{\xi} = \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}$)
- Information form can help recognize $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ of a Gaussian when doing algebraic manipulations

Multivariate Gaussian: The Covariance Matrix

The covariance matrix can be spherical, diagonal, or full



Picture courtesy: Computer vision: models, learning and inference (Simon Price)

Marginals and Conditionals from Gaussian Joint Distribution

- Given $\boldsymbol{x}$ having multivariate Gaussian distribution $\mathcal{N}(\boldsymbol{x}|\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with $\boldsymbol{\Lambda} = \boldsymbol{\Sigma}^{-1}$. Suppose

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{x}_a \\ \boldsymbol{x}_b \end{bmatrix}$$

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- Both results are extremely useful when working with Gaussian joint distributions

An Aside: Linear Transformations of Random Variables

- Suppose $\boldsymbol{x} = f(\boldsymbol{z}) = \mathbf{A}\boldsymbol{z} + \mathbf{b}$ be a linear function of an r.v. $\boldsymbol{z}$ (not necessarily Gaussian)
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Linear Gaussian Model

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- The following two distributions are of particular interest.

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- Exercise: Prove the above two results (MLAPP Chap. 4 and PRML Chap. 2 contain the proof)

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