Inference in Multiparameter Models, Conditional Posterior, Local Conjugacy

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Topics in Probabilistic Modeling and Inference (CS698X)

Feb 4, 2019

Today's Plan

- Foray into models with several parameters
- Goal will be to infer the posterior over all of them (not posterior for some, MLE-II for others)
- Idea of conditional/local posteriors in such problems
- Local conjugacy (which helps in computing conditional posteriors)
- Gibbs sampling (an algorithm that infer the joint posterior via conditional posteriors)
- An example: Bayesian matrix factorization (model with many parameters)
- Note: Conditional/local posterior, local conjugacy, etc are important ideas (will appear in many inference algorithms that we will see later)

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- What if we have a model with lots of parames/hyperparams and want posteriors over all of those?
  - Intractable in general but today we will look at a way of doing approx. inference in such models

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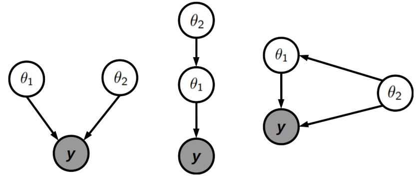
- Multiparameter models consist of two or more unknowns, say θ_1 and θ_2
- Given data $\boldsymbol{y}$, some examples for the simple two parameter case



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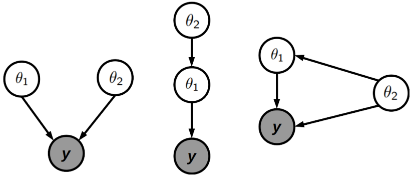

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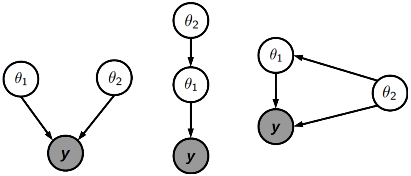

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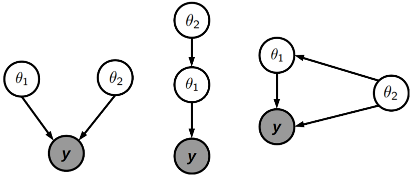

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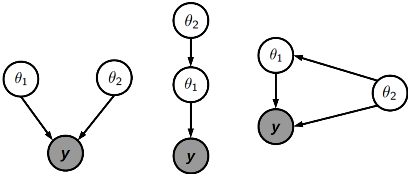

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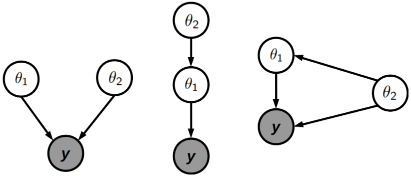

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  - Otherwise needs more work, e.g., MLE-II, MCMC, VB, etc. (already saw MLE-II, will see more later)

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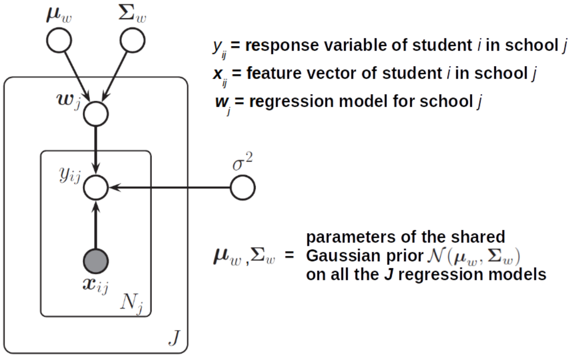

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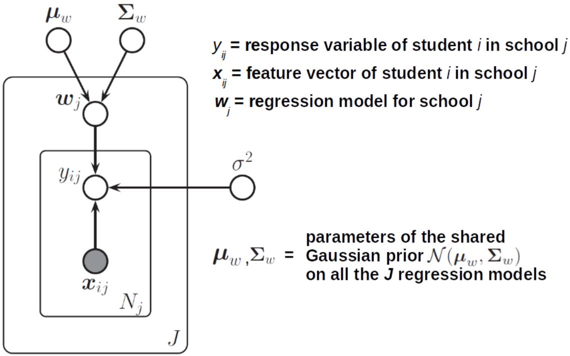
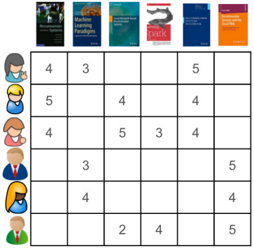


- .. and in fact, pretty much in any non-toy example of probablistic model :)

Another Example Problem: Matrix Factorization/Completion



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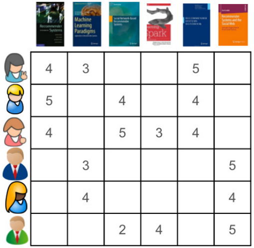

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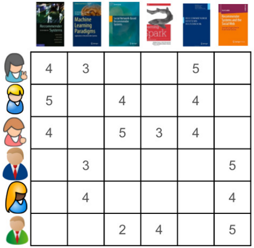

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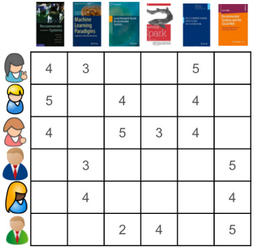

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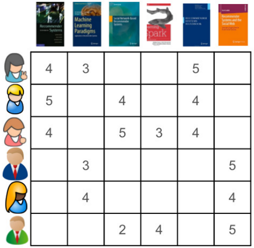

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- We would like to predict the unobserved values $r_{ij} \notin \mathcal{D}$

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- Given $\boldsymbol{u}_i$ and $\boldsymbol{v}_j$, any unobserved element in $\mathbf{R}$ can be predicted using the above

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- For simplicity, we will assume the hyperparams $\beta, \lambda_u, \lambda_v$ to be fixed and not to be learned

The Posterior

- Our target posterior distribution for this model will be

$$p(\mathbf{U}, \mathbf{V}|\mathbf{R}) = \frac{p(\mathbf{R}|\mathbf{U}, \mathbf{V})p(\mathbf{U}, \mathbf{V})}{\int \int p(\mathbf{R}|\mathbf{U}, \mathbf{V})p(\mathbf{U}, \mathbf{V})d\mathbf{U}d\mathbf{V}}$$

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- The denominator (and hence the posterior) is intractable!

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- Since CP of each var. depends on all other vars, inference algos based on computing CPs usually work in alternating fashion, until each CP converges (e.g., Gibbs sampling which we'll look at later)

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- One way to get the overall representation of the posterior can be can be using sampling based inference algorithms like Gibbs sampling or MCMC (more on this later)

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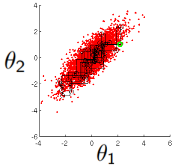
- The generated samples give a sample-based approximation of the multivariate distribution



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- More on Gibbs sampling when we discuss MCMC sampling algorithms (above is the high-level idea)

Back to Bayesian Matrix Factorization..

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- Note: the posterior of $\boldsymbol{u}_i$ doesn't actually depend on $\mathbf{U}_{-i}$ and rows of $\mathbf{R}$ except row i
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Bayesian Matrix Factorization

- The posterior predictive distribution for BMF (assuming other hyperparams known)

$$p(r_{ij}|\mathbf{R}) = \int\int p(r_{ij}|\boldsymbol{u}_i, \boldsymbol{v}_j)p(\boldsymbol{u}_i, \boldsymbol{v}_j|\mathbf{R})d\boldsymbol{u}_i d\boldsymbol{v}_j$$

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- In general, this is hard and needs approximation
  - If we are using Gibbs sampling, we can use the S samples $\{\boldsymbol{u}_i^{(s)}, \boldsymbol{v}_j^{(s)}\}_{s=1}^S$ to compute the mean
  - For the Gaussian likelihood case, the mean can be computed as

$$\mathbb{E}[r_{ij}] \approx \frac{1}{S}\sum_{s=1}^{S} \boldsymbol{u}_i^{(s)\top} \boldsymbol{v}_j^{(s)} \qquad \text{(Monte-Carlo averaging)}$$

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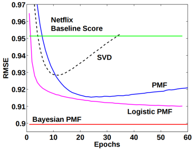
- A comparison of Bayesian MF with other methods (from Salakhutdinov and Mnih (2008))



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- It therefore helps to choose the likelihood model and priors on each param as exp. family distr.
  - Even if we can't get a globally conjugacy model, we can still get a model with local conjugacy
- Local conjugacy allows computing conditional posteriors that are needed in inference algos like Gibbs sampling, MCMC, EM, variational inference, etc.