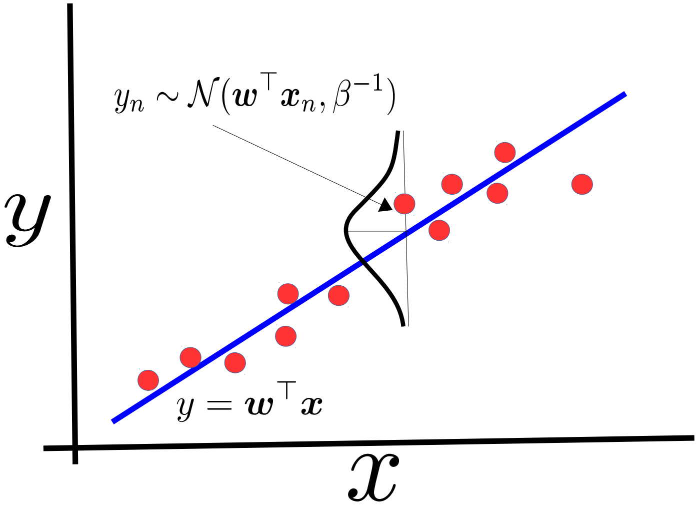
Probabilistic Models for Supervised Learning (Contd.)

Piyush Rai

Introduction to Machine Learning (CS771A)

August 21, 2018

Recap: Probabilistic Linear Regression



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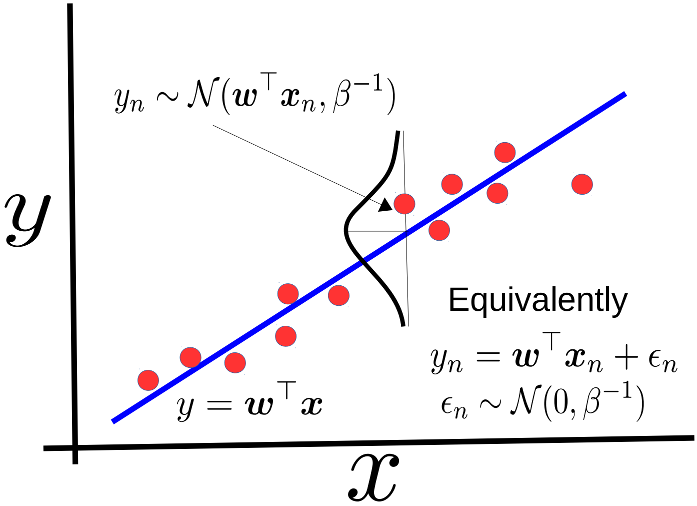

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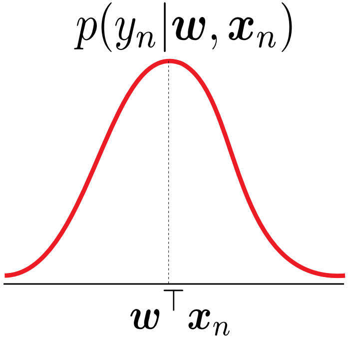

The Likelihood

$$p(y_n|\boldsymbol{w}, \boldsymbol{x}_n) = \mathcal{N}(y_n|\boldsymbol{w}^\top \boldsymbol{x}_n, \beta^{-1}) = \sqrt{\frac{\beta}{2\pi}} \exp\left[-\frac{\beta}{2}(y_n - \boldsymbol{w}^\top \boldsymbol{x}_n)^2\right]$$

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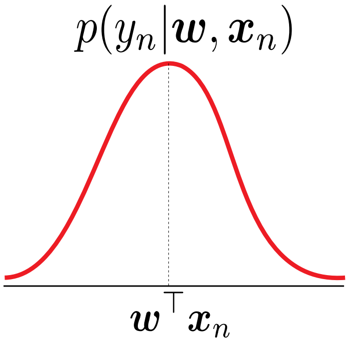

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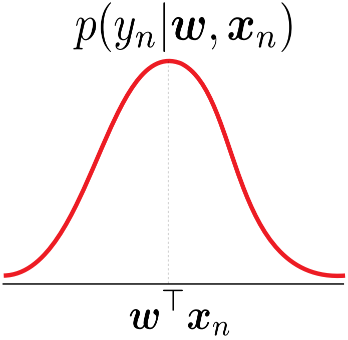

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$$\left(\frac{\beta}{2\pi}\right)^{N/2} \exp\left[-\frac{\beta}{2}\sum_{n=1}^{N}(y_n - \boldsymbol{w}^\top \boldsymbol{x}_n)^2\right] = \frac{1}{\sqrt{\det(2\pi\beta^{-1}\mathbf{I}_N)}} \exp\left[-\frac{\beta}{2}(\boldsymbol{y} - \mathbf{X}\boldsymbol{w})^\top(\boldsymbol{y} - \mathbf{X}\boldsymbol{w})\right]$$

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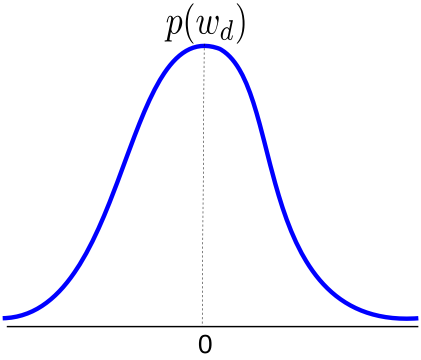

The Prior

$$p(w_d) = \mathcal{N}(w_d|0, \lambda^{-1}) = \sqrt{\frac{\lambda}{2\pi}} \exp\left[-\frac{\lambda}{2} w_d^2\right]$$

Zero-mean Gaussian prior encourages weights to be small. Precision λ controls how strong this prior is.

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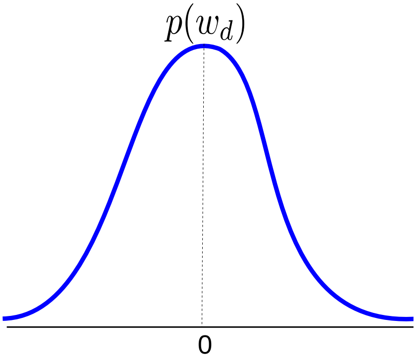

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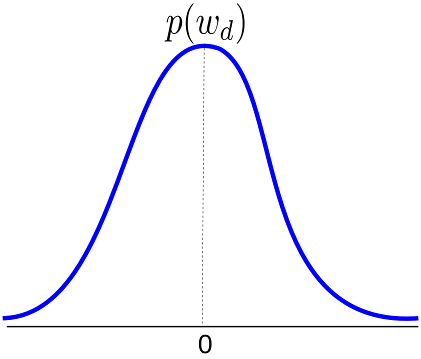

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Recap: MLE, MAP, and Bayesian Inference for Prob. Lin. Reg.

- For MLE, we maximize the log-likelihood. Ignoring constants w.r.t. $\boldsymbol{w}$, we have

$$\hat{\boldsymbol{w}}_{MLE} = \arg\max_{\boldsymbol{w}} \log p(\boldsymbol{y}|\mathbf{X}, \boldsymbol{w})$$

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- For Bayesian inference, we compute the full posterior. Easily computable (thanks to conjugacy)

$$p(\boldsymbol{w}|\boldsymbol{y}, \mathbf{X}) = \mathcal{N}(\boldsymbol{\mu}_N, \boldsymbol{\Sigma}_N)$$

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Recap: Predictive Distribution for Prob. Lin. Reg.

- When using MLE/MAP estimate of $\boldsymbol{w}$, we compute the "plug-in" predictive distribution

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p(y_*|\boldsymbol{x}_*, \mathbf{X}, \boldsymbol{y}) &\approx p(y_*|\boldsymbol{x}_*, \boldsymbol{w}_{MLE}) = \mathcal{N}(\boldsymbol{w}_{MLE}^\top \boldsymbol{x}_*, \beta^{-1}) \\
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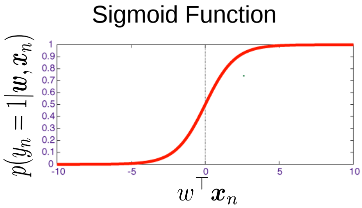
- For Bayesian approach, mean of predicted output is $\boldsymbol{w}_N^\top \boldsymbol{x}_*$, variance is $\beta^{-1} + \boldsymbol{x}_*^\top \boldsymbol{\Sigma}_N \boldsymbol{x}_*$(note the different variance for each test input, unlike MLE/MAP prediction)

Recap: Logistic Regression

- Logistic Regression models $p(y_n = 1|\boldsymbol{w}, \boldsymbol{x}_n)$ using the sigmoid function

$$p(y_n = 1|\boldsymbol{w}, \boldsymbol{x}_n) = \mu_n = \sigma(\boldsymbol{w}^\top \boldsymbol{x}_n) = \frac{1}{1 + \exp(-\boldsymbol{w}^\top \boldsymbol{x}_n)} = \frac{\exp(\boldsymbol{w}^\top \boldsymbol{x}_n)}{1 + \exp(\boldsymbol{w}^\top \boldsymbol{x}_n)}$$



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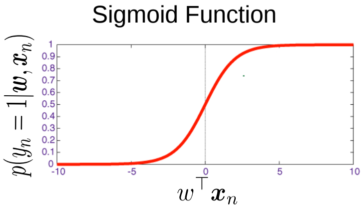

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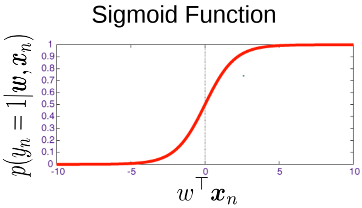

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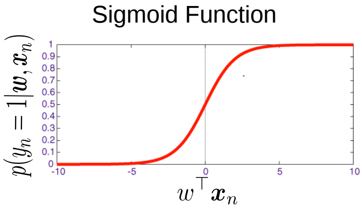

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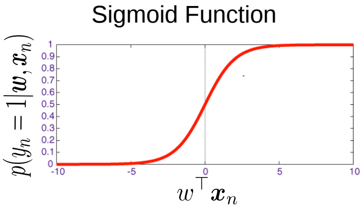

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- Can also use a Gaussian prior $p(\boldsymbol{w}) = \mathcal{N}(\boldsymbol{w}|\mathbf{0}, \lambda^{-1}\mathbf{I}_D)$ just like in probabilistic linear regression
- Can estimate $\boldsymbol{w}$ via MLE, MAP, or (a somewhat hard to do) fully Bayesian inference

Recap: Logistic Regression

- Logistic regression can be extended to more than 2 classes

$$p(y_n = k|\boldsymbol{x}_n, \mathbf{W}) = \frac{\exp(\boldsymbol{w}_k^\top \boldsymbol{x}_n)}{\sum_{\ell=1}^K \exp(\boldsymbol{w}_\ell^\top \boldsymbol{x}_n)} = \mu_{nk}$$

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- MLE/MAP for logistic/softmax does not have closed form solution (unlike linear regression case)

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Generative Models for Supervised Learning

$$p(y|x) = \frac{p(x, y)}{p(x)}$$

Here, we will model both inputs and outputs!

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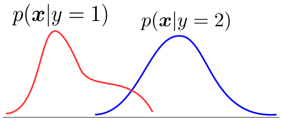
  - Another exercise: Try to derive the MAP estimate of $\boldsymbol{\pi}$ and also the full posterior (good news: multinoulli and Dirichlet are conjugate to each other, so full posterior is easy)

Generative Classification: Estimating Class-Conditional Distr.



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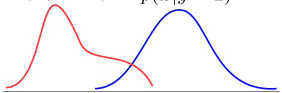

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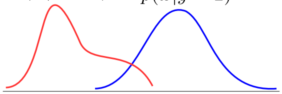

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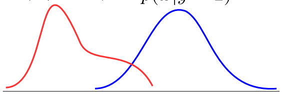

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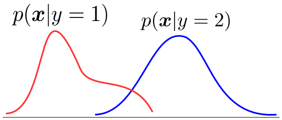

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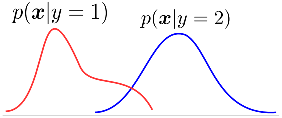

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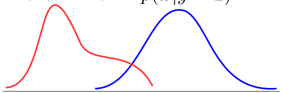

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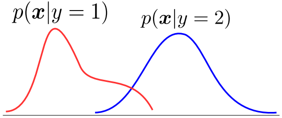

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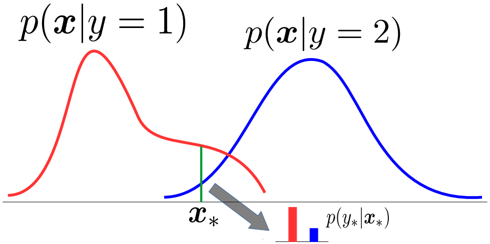
- If $p(y=k)$ is the same for all the classes then, we simple compare $p(\boldsymbol{x}|y=k)$



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- Assume each class-conditional to be a Gaussian

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- If using MLE/MAP estimate of θ, the predictive distribution will be

$$p(y_* = k|\boldsymbol{x}_*, \theta) = \frac{\pi_k|\boldsymbol{\Sigma_k}|^{-1/2}\exp\left[-\frac{1}{2}(\mathbf{x}_* - \boldsymbol{\mu}_k)^\top\boldsymbol{\Sigma_k^{-1}}(\mathbf{x}_* - \boldsymbol{\mu}_k)\right]}{\sum_{k=1}^K \pi_k|\boldsymbol{\Sigma_k}|^{-1/2}\exp\left[-\frac{1}{2}(\mathbf{x}_* - \boldsymbol{\mu}_k)^\top\boldsymbol{\Sigma_k^{-1}}(\mathbf{x}_* - \boldsymbol{\mu}_k)\right]}$$

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- The generative classification prediction rule we saw had

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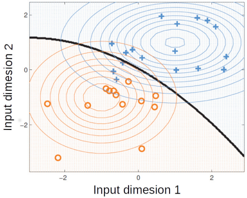

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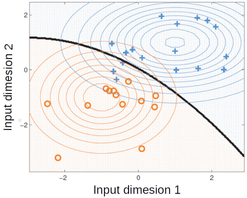

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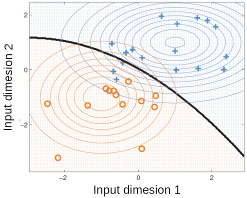

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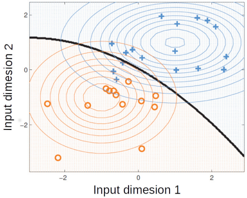

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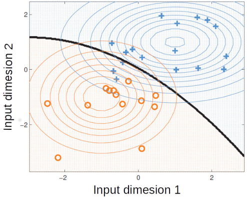

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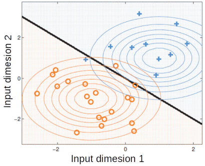

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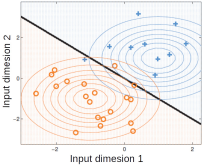

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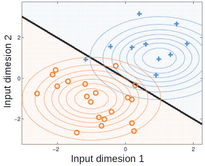

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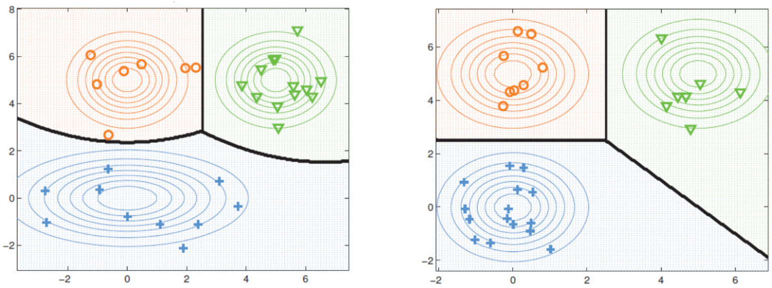
Decision Boundaries

- Depending on the form of the covariance matrices, the boundaries can be quadratic/linear



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- Interestingly, this has exactly the same form as the softmax classification model (saw it in last class), which is a discriminative model, as opposed to a generative model.

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- This is equivalent to assigning $\boldsymbol{x}$ to the "closest" class in terms of a Mahalanobis distance

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$$\hat{y} = \arg\max_k p(y = k|\mathbf{x}) = \arg\max_k \; \pi_k \exp\left[-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu}_k)^\top \boldsymbol{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu}_k)\right]$$

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- Further, let's assume the classes to be of equal size, i.e., $\pi_k = 1/K$. Then we will have

$$\hat{y} = \arg\min_k \; (\mathbf{x} - \boldsymbol{\mu}_k)^\top \boldsymbol{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu}_k)$$

- This is equivalent to assigning $\boldsymbol{x}$ to the "closest" class in terms of a Mahalanobis distance
  - The covariance matrix "modulates" how the distances are computed

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- Generative models are also useful for unsupervised and semi-supervised learning (will look at later)

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- A good density estimation model is necessary for generative classification model to work well

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- Note: Can use generative models for doing regression as well (will be an exercise)