Learning via Probabilistic Modeling

Piyush Rai

Introduction to Machine Learning (CS771A)

August 14, 2018

Recap: Linear Models and Linear Regression

- Linear model: Each output is a linearly weighted combination of the inputs

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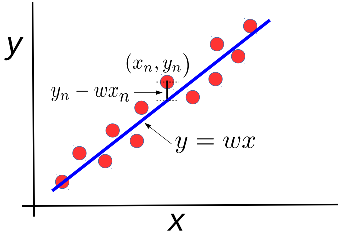

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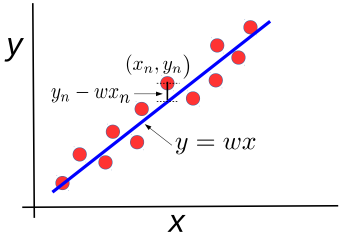

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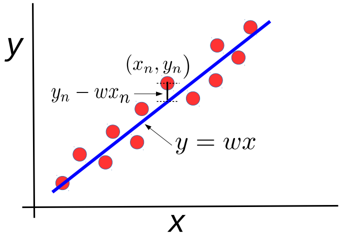

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- This requires optimizing the loss w.r.t. $\boldsymbol{w}$

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  - $R(.)$ denotes the regularizer chosen to make f simple (e.g., ℓ_2 regularization)

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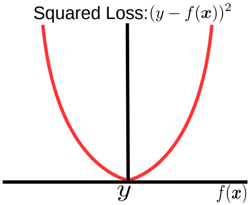

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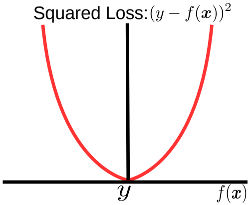

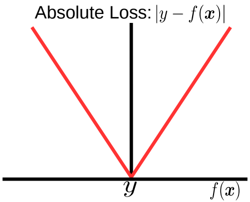

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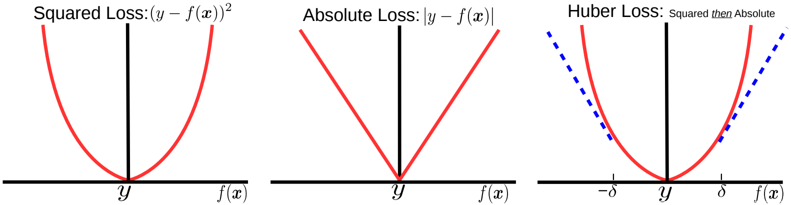

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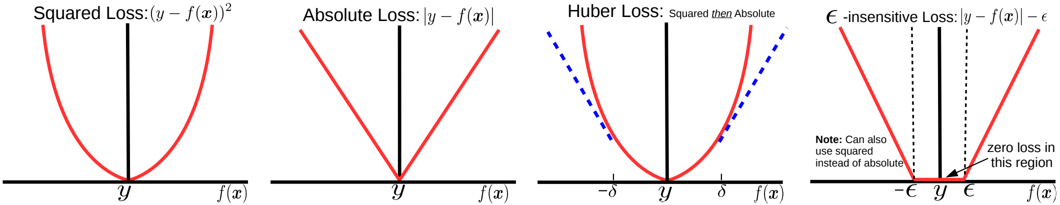

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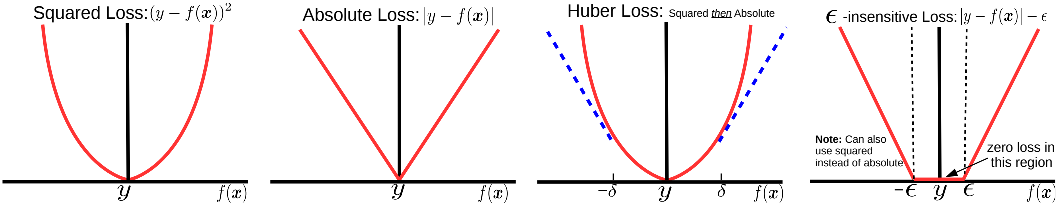

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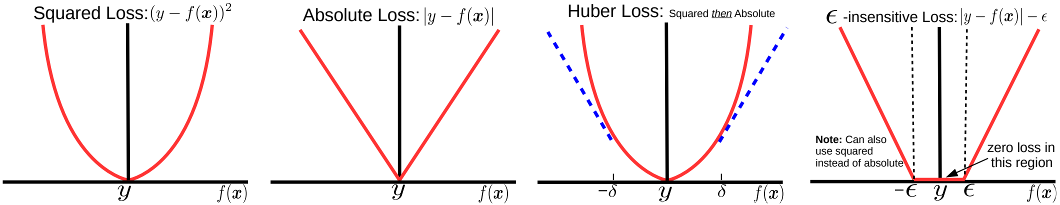

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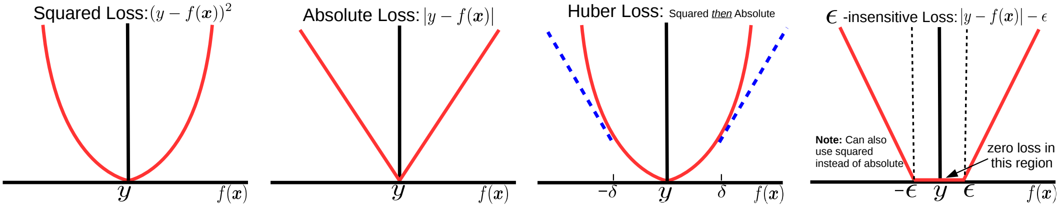

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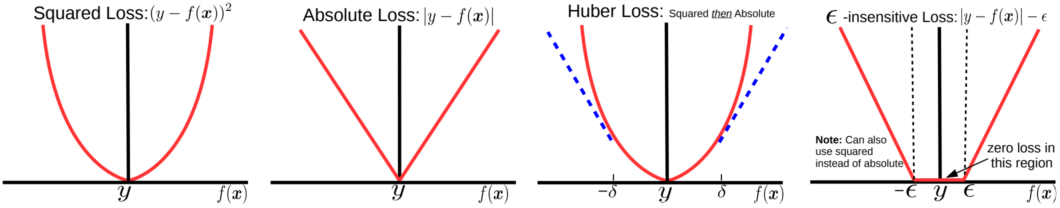

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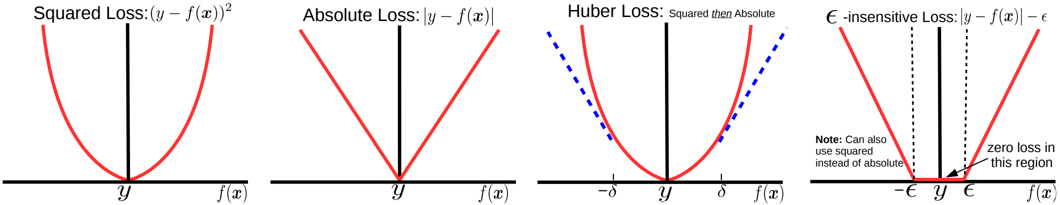

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- Will revisit many of these aspects when we talk about optimization techniques for ML

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- Some common examples of such assumptions
  - Classes are separable by a large margin
  - The function is "smooth"
  - Only a few features are relevant for the prediction

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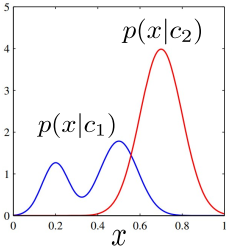

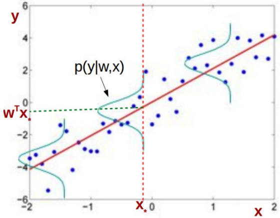

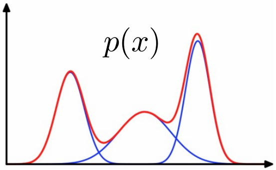

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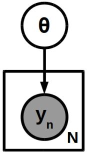

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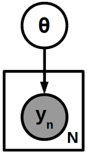

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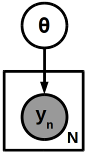

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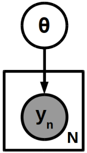

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  - Prediction: Compute predictive distribution $p(y_*|\boldsymbol{y})$ for new data (or mean/variance of $p(y_*|\boldsymbol{y})$)

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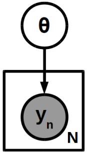

Plate Notation

- Some of the things we may be interested in
  - Parameter Estimation: Estimate θ given the observed data $\boldsymbol{y}$
  - Prediction: Compute predictive distribution $p(y_*|\boldsymbol{y})$ for new data (or mean/variance of $p(y_*|\boldsymbol{y})$)
- **Important:** Pretty much any ML problem (sup/unsup) can be formulated like this

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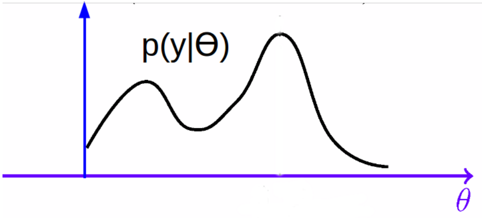

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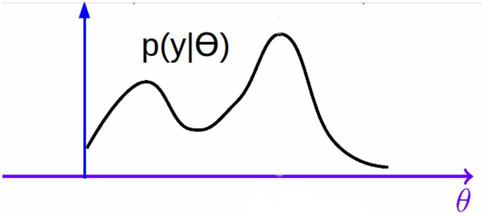

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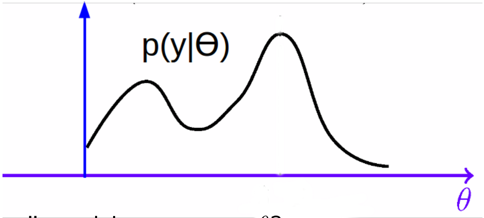

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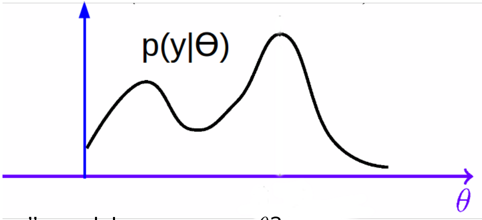

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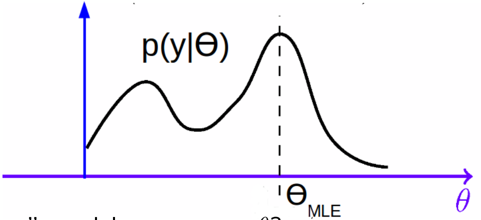

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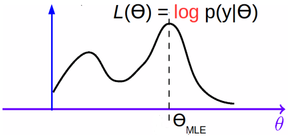

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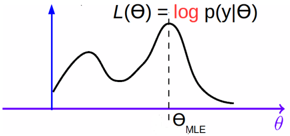

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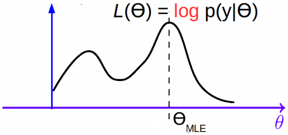

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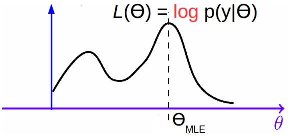

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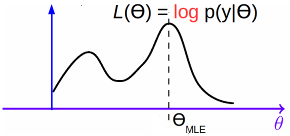

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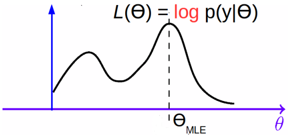

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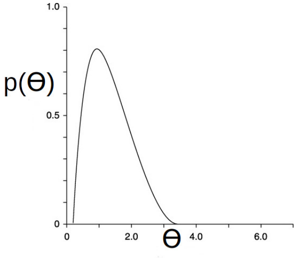
- In probabilistic models, we can specify a prior distribution $p(\theta)$ on parameters



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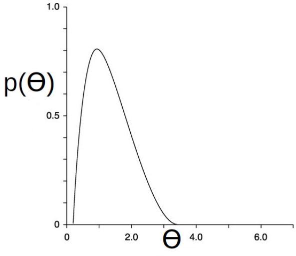

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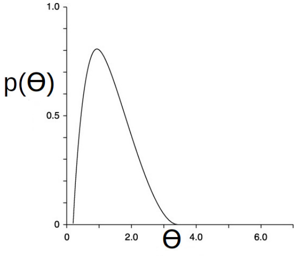

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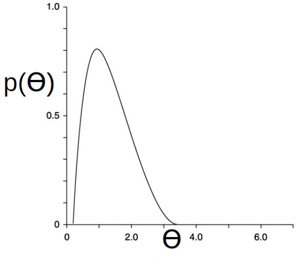

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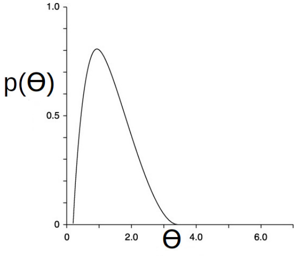

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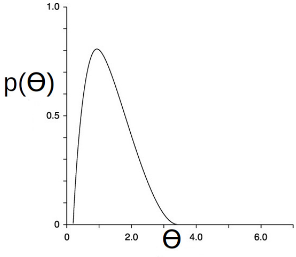

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- Note: A uniform prior distribution is the same as using no prior!

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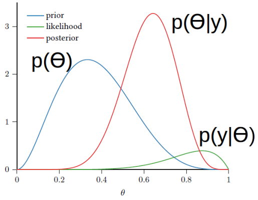
- We can **combine** the prior $p(\theta)$ with the likelihood $p(\boldsymbol{y}|\theta)$ using Bayes rule and define the posterior distribution over the parameters θ

$$p(\theta|\boldsymbol{y}) = \frac{p(\boldsymbol{y}|\theta)p(\theta)}{p(\boldsymbol{y})}$$



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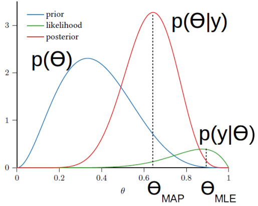

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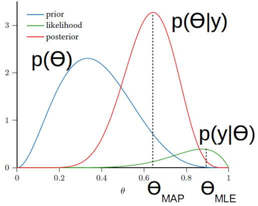

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- Note that the prior sort of "pulls" θ_{MLE} toward's the prior distribution's mean/mode

Maximum-a-Posteriori (MAP) Estimation

- We will work with the **log** posterior probability (it is easier)

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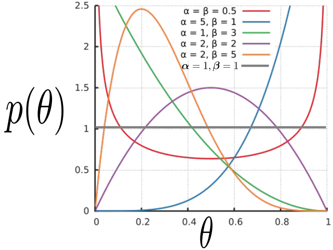

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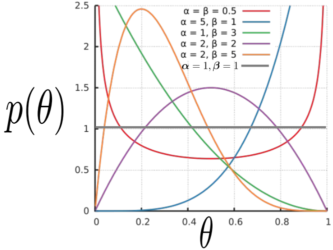

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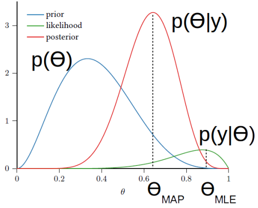
Inferring the Full Posterior (a.k.a. Fully Bayesian Inference)

- MLE/MAP only give us a **point estimate** of θ. Doesn't capture the uncertainty in θ



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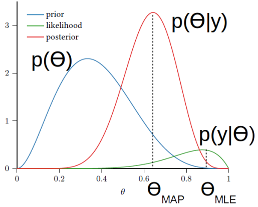

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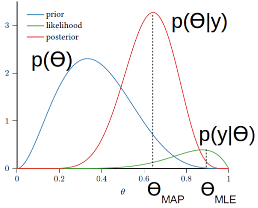

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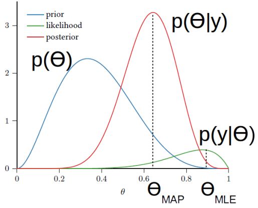

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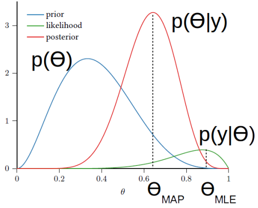

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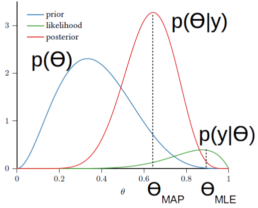

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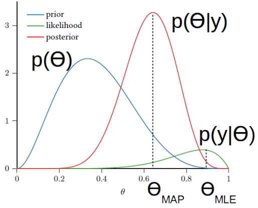

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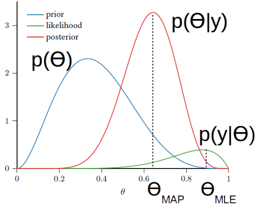

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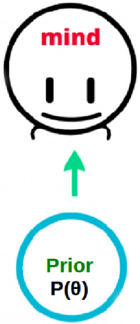
- Fully Bayesian inference fits naturally into an "online" learning setting



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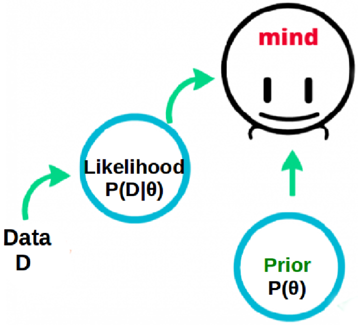

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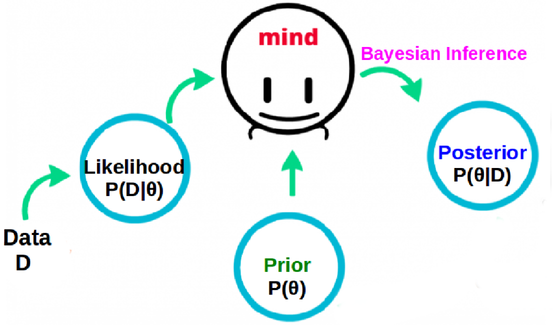

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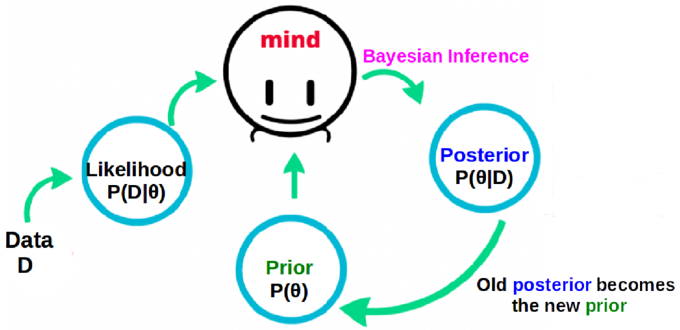

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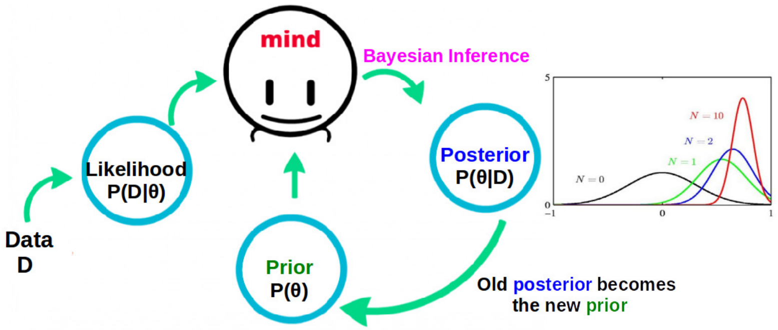

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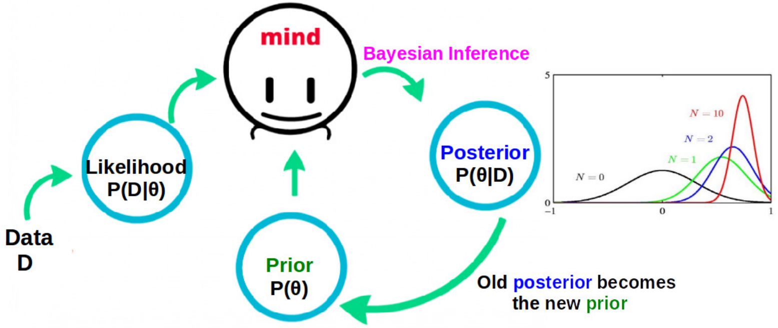

- Our belief about θ keeps getting updated as we see more and more data

Fully Bayesian Inference: An Example

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- With Bernoulli likelihood and Beta prior (a conjugate pair), the posterior is also Beta (exercise)

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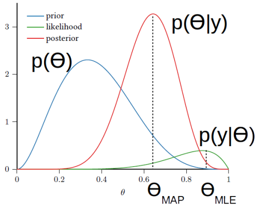
- Can verify the above by simply plugging in the expressions of likelihood and prior into the Bayes rule and identifying the form of resulting posterior (note: this may not always be easy)



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- Thus for this example (where observations are assumed to come from a Bernoulli)

MLE prediction: $p(y_{N+1} = 1|\boldsymbol{y})$

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- Note that the fully Bayesian approach to prediction averages over all possible values of θ, weighted by their respective posterior probabilities (easy in this example, but a hard problem in general)

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