Reinforcement Learning

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Introduction to Machine Learning (CS771A)

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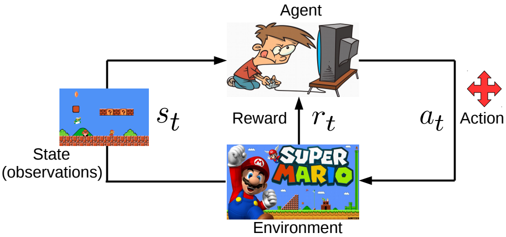
Reinforcement Learning

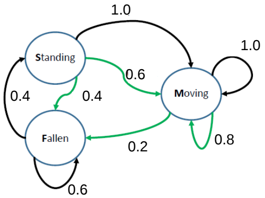
- Supervised Learning: Uses explicit supervision (input-output pairs)
- In many learning problems that need supervision, it is hard to provide explicit supervision
- Example: Learning a policy/strategy of an agent acting in an environment



- Agent lives in states, takes actions, gets rewards, and goal is to maximize its total reward
- Reinforcement Learning (RL) is a way to solve such problems
- Many applications: Robotics, autonomous driving, computer game playing, online advertising, etc.

Markov Decision Processes (MDP)

- MDP gives a formal way to define RL problems
- An MDP consists of a tuple $(S, A, \{P_{sa}\}, \gamma, R)$
- S is a set of states (discrete or continuous valued)
- A is a set of actions
- P_{sa} is a probability distribution over possible states
  - $P_{sa}(s')$: Prob. of switching to s' if we took action a in s
  - For discrete states, P_{sa} is an $|S|$ length probability vector
  - Another notation for $P_{sa}(s')$: $T(s, a, s')$
- $R : S \times A \mapsto \mathbb{R}$ is the reward function
  - Reward for reaching state s: $R(s, a)$
- $\gamma \in [0, 1)$ is called discount factor for future rewards
- P_{sa} and R may be unknown (may need to be learned)



slow action (black)
fast action (green)

States = [Standing, Moving, Fallen]

Actions = [Slow, Fast]

T(Standing, Slow, Moving) = 1
T(Standing, Fast, Moving) = 0.6
T(Standing, Fast, Fallen) = 0.4
...

Payoff and Expected Payoff

- Payoff defines the cumulative reward
- Upon visiting states $s_0, s_1, \ldots$ with actions $a_0, a_1, \ldots$, the payoff:

$$R(s_0, a_0) + \gamma R(s_1, a_1) + \gamma^2 R(s_2, a_2) + \ldots$$

- Reward at time t is discounted by γ^t (note: $\gamma < 1$)
  - We care more about immediate rewards, rather than the future rewards
- If rewards defined in terms of states only, then the payoff:

$$R(s_0) + \gamma R(s_1) + \gamma^2 R(s_2) + \ldots$$

- We want to choose actions over time (by learning a "policy") to maximize the expected payoff

$$\mathbb{E}[R(s_0) + \gamma R(s_1) + \gamma^2 R(s_2) + \ldots]$$

- The expectation $\mathbb{E}$ is w.r.t. all possibilities of the initial state s_0

Policy Function and Value Function

- Policy Function or **Policy** is a function $\pi : S \mapsto A$, mapping from states to actions
- For an agent with policy π, the action in state s: $a = \pi(s)$. Want to learn the best π
- For any policy π, we can define the **Value Function**

$$V^\pi(s) = \mathbb{E}[R(s_0) + \gamma R(s_1) + \gamma^2 R(s_2) + \ldots | s_0 = s, \pi]$$

- $V^\pi(s)$ is the expected payoff starting in state s **and** following policy π
- For finite state spaces, $V^\pi(s)$ will be a vector of size $|S|$
- **Bellman's Equation:** Gives a recursive definition of the above Value Function:

$$\begin{aligned} V^\pi(s) &= R(s) + \gamma \sum_{s' \in S} P_{s\pi(s)}(s') \times V^\pi(s') \\ &= R(s) + \gamma \mathbb{E}_{s' \sim P_{s\pi(s)}}[V^\pi(s')] \end{aligned}$$

- It's the immediate reward + expected sum of future discounted rewards

Computing the Value Function

- Given π, Bellman's equation can be used to compute the value function $V^\pi(s)$

$$V^\pi(s) = R(s) + \gamma \sum_{s' \in S} P_{s\pi(s)}(s')V^\pi(s')$$

- For finite-state MDP, it gives us $|S|$ equations with $|S|$ unknowns $\Rightarrow$ Efficiently solvable
- The **Optimal Value Function** is defined as

$$V^*(s) = \max_\pi V^\pi(s)$$

- It's the best possible expected payoff that any policy π can give
- The Optimal Value Function can also be defined as:

$$\boxed{V^*(s) = R(s) + \max_{a \in A} \gamma \sum_{s' \in S} P_{sa}(s')V^*(s')}$$

Optimal Policy

- The **Optimal Value Function**:

$$V^*(s) = R(s) + \max_{a \in A} \gamma \sum_{s' \in S} P_{sa}(s') V^*(s')$$

- Given the optimal value function V^*, the **Optimal Policy** $\pi^* : S \mapsto A$:

$$\boxed{\pi^*(s) = \arg\max_{a \in A} \sum_{s' \in S} P_{sa}(s') V^*(s')} \quad (1)$$

- $\pi^*(s)$ gives the action a that maximizes the optimal value function for that state
- Three popular methods to find the optimal policy
  - **Value Iteration:** Estimate V^* and then use Eq 1
  - **Policy Iteration:** Iterate between learning the optimal policy π^* and learning V^*
  - **Q Learning** (a variant of value iteration)

Finding the Optimal Policy: Value Iteration

- Iteratively compute the optimal value function V^* as follows
  - For each state s, initialize $V(s) = 0$
  - Repeat until convergence
    - For each state s, update $V(s)$ as

$$V(s) = R(s) + \max_{a \in A} \sum_{s'} P_{sa}(s')V(s')$$

- Value Iteration property: V converges to V^*
- Upon convergence, use $\pi^*(s) = \arg\max_{a \in A} \sum_{s' \in S} P_{sa}(s')V^*(s')$
- **Note:** The inner loop can update $V(s)$ for all states simultaneously, or in some order

Finding the Optimal Policy: Policy Iteration

- Iteratively compute the policy π until convergence
  - Initialize π randomly
  - Repeat until convergence
    1. Let $V = V^\pi$
    2. For each state s, set $\pi(s) = \arg\max_{a \in A} \sum_{s'} P_{sa}(s')V(s')$
- Step (1) the computes the value function for the current policy π
  - Can be done using Bellman's equations (solving $|S|$ equations in $|S|$ unknowns)
- Step (2) gives the policy that is greedy w.r.t. V

Finding the Optimal Policy: Q Learning

- This is a variant of value iteration
- However, instead of iterating over V, we iterate over a "Q function"
- Recall the optimal value function

$$V^*(s) = R(s) + \max_{a \in A} \gamma \sum_{s' \in S} P_{sa}(s') V^*(s')$$

- Define $Q(s,a) = R(s) + \gamma \sum_{s' \in S} P_{sa}(s') V^*(s')$ then

$$V^*(s) = \max_{a \in A} Q(s,a)$$

- We can iteratively learn $Q(s,a)$ until convergence

$$Q_{t+1}(s,a) = R(s) + \gamma \sum_{s' \in S} P_{sa}(s') \max_{a' \in A} Q_t(s,a')$$

- Then set $V^*(s) = \arg\max_{a \in A} Q(s,a)$

Learning an MDP Model

- So far we assumed:
  - State transition probabilities $\{P_{sa}\}$ are given
  - Rewards $R(s)$ at each state are known
- Often we don't know these and want to learn these
- These are learned using experience (i.e., a set of previous trials)

$$s_0^{(1)} \xrightarrow{a_0^{(1)}} s_1^{(1)} \xrightarrow{a_1^{(1)}} s_2^{(1)} \xrightarrow{a_2^{(1)}} s_3^{(1)} \xrightarrow{a_3^{(1)}} \ldots$$
$$s_0^{(2)} \xrightarrow{a_0^{(2)}} s_1^{(2)} \xrightarrow{a_1^{(2)}} s_2^{(2)} \xrightarrow{a_2^{(2)}} s_3^{(2)} \xrightarrow{a_3^{(2)}} \ldots$$
$$\ldots$$

- $s_i^{(j)}$ is the state at time i of trial j
- $a_i^{(j)}$ is the corresponding action at that state

Learning an MDP Model

- Maximum likelihood estimate of state transition probabilities:

$$P_{sa}(s') = \frac{\#\text{ of times we took action } a \text{ in state } s \text{ and got to } s'}{\#\text{ of times we took action } a \text{ in state } s}$$

- Note: if action a is never taken in state s, the above ratio is $0/0$
  - In that case: $P_{sa}(s') = 1/|S|$ (uniform distribution over all states)
- P_{sa} is easy to update if we gather more experience (i.e., do more trials)
  - .. just add counts in the numerator and denominator
- Likewise, the expected reward $R(s)$ in state s can be computed
  - $R(s)$ = average reward in state s across all the trials

MDP Learning + Policy Learning

Alternate between learning the MDP (P_{sa} and R), and learning the policy

Policy learning step can be done using value iteration or policy iteration

The Algorithm (assuming value iteration)

- Randomly initialize policy π
- Repeat until convergence
  1. Execute policy π in the MDP to generate a set of trials
  2. Use this "experience" to estimate P_{sa} and R
  3. Apply value iteration with the estimated P_{sa} and R

     $\Rightarrow$ Gives a new estimate of the value function V
  4. Update policy π as the greedy policy w.r.t. V

Note: Step 3 can be made more efficient by initializing V with values from the previous iteration

Value Iteration vs Policy Iteration

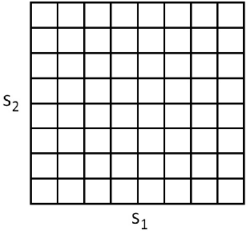
- Small state spaces: Policy Iteration typically very fast and converges quickly
- Large state spaces: Policy Iteration may be slow
  - Reason: Policy Iteration needs to solve a large system of linear equations
  - Value iteration is preferred in such cases
- Very large state spaces: Value function can be *approximated* using some regression algorithm
  - Optimality guarantee is lost however

MDP with Continuous State Spaces

- A car moving in 2D: $s = (x, y, \theta, \dot{x}, \dot{y}, \dot{\theta})$. Thus $S = \mathbb{R}^6$
- Helicopter flying in 3D: $s = (x, y, z, \phi, \theta, \psi, \dot{x}, \dot{y}, \dot{z}, \dot{\phi}, \dot{\theta}, \dot{\psi})$. Thus $S = \mathbb{R}^{12}$
- In general, the state space could be infinite $S = \mathbb{R}^n$
- How to handle these continuous state spaces?

Discretization

- Suppose the state space is 2D: $s = (s_1, s_2)$. Can discretize it



- Call each discrete state $\bar{s}$, discretized state space $\bar{S}$, and define the MDP as

$$(\bar{S}, A, \{P_{\bar{s}a}\}, \gamma, R)$$

- Can now use value iteration or policy itearation on this discrete state space
- Limitations? Piecewise constant V^* and π^* (isn't realistic). Doesn't work well in high-dim state spaces (resulting discrete space space too huge)
- Discretization usually done only for 1D or 2D state-spaces

Policy Learning in Continuous State Spaces

- Policy learning requires learning the value function V^*
- Can we do away with discretization and approximate V^* directly?
- To do so, we will need (an approximate) model of the underlying MDP

Approximating the MDP Model

- Execute a set of trials

$$s_0^{(1)} \xrightarrow{a_0^{(1)}} s_1^{(1)} \xrightarrow{a_1^{(1)}} s_2^{(1)} \xrightarrow{a_2^{(1)}} \cdots \xrightarrow{a_{T-1}^{(1)}} s_T^{(1)}$$
$$s_0^{(2)} \xrightarrow{a_0^{(2)}} s_1^{(2)} \xrightarrow{a_1^{(2)}} s_2^{(2)} \xrightarrow{a_2^{(2)}} \cdots \xrightarrow{a_{T-1}^{(2)}} s_T^{(2)}$$
$$\cdots$$
$$s_0^{(m)} \xrightarrow{a_0^{(m)}} s_1^{(m)} \xrightarrow{a_1^{(m)}} s_2^{(m)} \xrightarrow{a_2^{(m)}} \cdots \xrightarrow{a_{T-1}^{(m)}} s_T^{(m)}$$

- Use this data to learn a function that predicts s_{t+1} given s_t and a, e.g.,

$$s_{t+1} = As_t + Ba_t$$
$$\arg\min_{A,B} \sum_{i=1}^{m} \sum_{t=0}^{T-1} \left\| s_{t+1}^{(i)} - \left(As_t^{(i)} + Ba_t^{(i)} \right) \right\|^2$$

- A and B can be estimate from the trial data
- Can also make the function stochastic/noisy: $s_{t+1} = As_t + Ba_t + \epsilon_t$ where $\epsilon_t \sim \mathcal{N}(0, \Sigma)$ is the random noise (Σ can also be learned)

Approximating the MDP Model

- Can also learn nonlinear functions $s_{t+1} = f(s_t)$

$$s_{t+1} = A\phi_s(s_t) + B\phi_a(a_t)$$

- Any nonlinear regression algorithm can be used here

Approximating the Value Function

- We will use “Fitted Value Iteration” methods
- Recall the value iteation

$$
\begin{aligned}
V(s) &:= R(s) + \gamma \max_a \int_{s'} P_{sa}(s')V(s')ds' \\
&= R(s) + \gamma \max_a \mathrm{E}_{s' \sim P_{sa}}[V(s')]
\end{aligned}
$$

- Note: sum replaced by integral (since the state space S is continuous)
- Want a model for $V(s)$. Let's assume $V(s) = \theta^\top \phi(s)$
- We would need some training data in order to learn θ

$$\{V(s^i), \phi(s^i)\}_{i=1}^m$$

- We will generate such training data and learn θ in an alternating fashion

Fitted Value Iteration: The Full Algorithm

1. Randomly sample m states $s^{(1)}, s^{(2)}, \ldots s^{(m)} \in S$.

2. Initialize $\theta := 0$.

3. Repeat {

    For $i = 1, \ldots, m$ {

        For each action $a \in A$ {

            Sample $s'_1, \ldots, s'_k \sim P_{s^{(i)}a}$ (using a model of the MDP).

            Set $q(a) = \frac{1}{k}\sum_{j=1}^{k} R(s^{(i)}) + \gamma V(s'_j)$

                // Hence, $q(a)$ is an estimate of $R(s^{(i)}) + \gamma \mathrm{E}_{s' \sim P_{s^{(i)}a}}[V(s')]$.

        }

        Set $y^{(i)} = \max_a q(a)$.

            // Hence, $y^{(i)}$ is an estimate of $R(s^{(i)}) + \gamma \max_a \mathrm{E}_{s' \sim P_{s^{(i)}a}}[V(s')]$.

    }

    // In the original value iteration algorithm (over discrete states)

    // we updated the value function according to $V(s^{(i)}) := y^{(i)}$.

    // In this algorithm, we want $V(s^{(i)}) \approx y^{(i)}$, which we'll achieve

    // using supervised learning (linear regression).

    Set $\theta := \arg\min_\theta \frac{1}{2}\sum_{i=1}^{m}\left(\theta^T \phi(s^{(i)}) - y^{(i)}\right)^2$

Fitted Value Iteration

- Other nonlinear regression algorithms can also be used

$$V(s) = f(\phi(s))$$

  where f is a nonlinear function (e.g., modeled by a Gaussian Process)

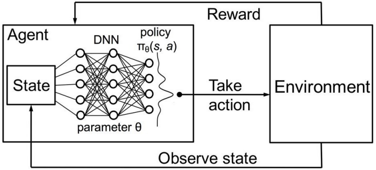
- Note: Fitted value iteration is not guaranteed to converge (though, in practice, mostly it does)
- The final output is V (an approximation to V^*)
- V implicitly represents our policy π. The optimal action

$$\arg\max_a \mathbb{E}_{s' \sim P_{sa}}[V(s')]$$

Other Topics related to RL

- Inverse Reinforcement Learning (IRL)
    - Doesn't assume the reward function to be known. Learns it
- Imitation Learning: Imitate an demonstrator/demonstrations

- Deep Reinforcement Learning



Summary

- Basic introduction to Reinforcement Learning
- Looked at the definition of a Markov Decision Process (MDP)
- Looked at methods for learning the MDP parameters from data
  - Easily and exactly for the discrete state-space case
  - Using function approximation methods in the continuous case
- Looked at methods for Policy Learning
  - MDP Learning and Policy Learning usually done jointly