Learning to Recommend via Matrix Factorization/Completion

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Introduction to Machine Learning (CS771A)

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- A notion of relevance: I will buy/watch/like items that are similar to ones I did in the past
- Has been a very active research topic (in ML and allied areas) for a long time
  - Even a dedicated conference focusing on this topic specifically - RecSys

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4	4	?	?	?	3

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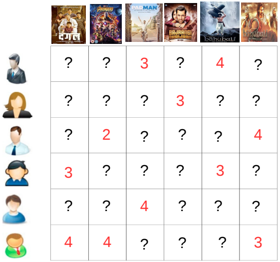

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- Note: In addition to the user-item matrix, we may have additional info about the user/items
  - Some examples: User meta-data, item content description, user-user network, item-item similarity, etc.

Some Notation

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A Simple Heuristic: Item based "Collaborative Filtering"

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- For each user-item pair (n, m), compute the missing rating X_{nm} as

$$X_{nm} \approx \frac{1}{|\Omega_{r_n}|} \sum_{m' \in \Omega_{r_n}} S^{(I)}_{mm'} X_{nm'}$$

where $S^{(I)}_{mm'} \in (0, 1)$ is the similarity between items m and m' (suppose known)

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Limitations of Item/User Based Approach

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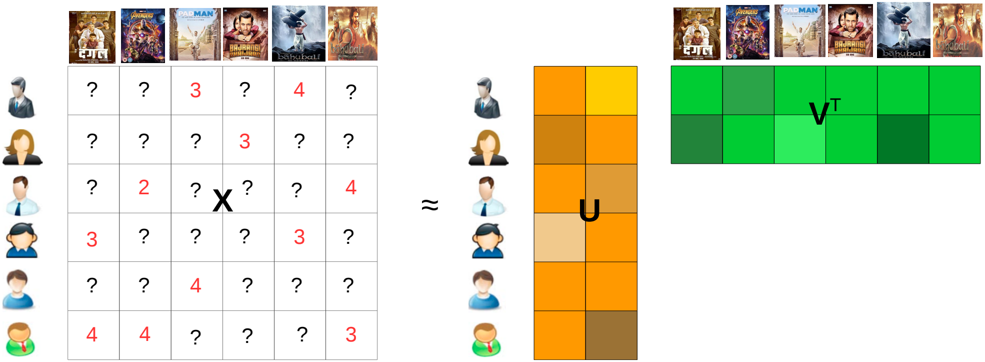
- User-User or Item-Item similarities may not be known beforehand
- We may have very little data in the user-item matrix and averaging may not be reliable

Towards a better approach: Matrix Factorization



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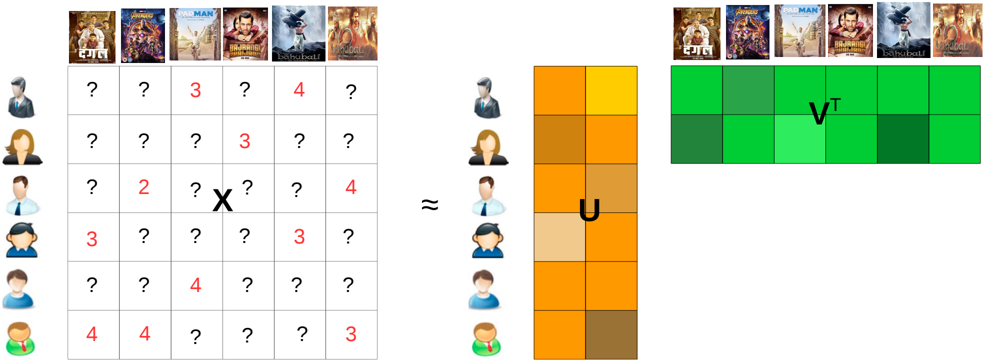

If we can do the above factorization then any missing $X_{nm} \approx \boldsymbol{u}_n^\top \boldsymbol{v}_m$

Matrix Factorization

- Given a matrix $\mathbf{X}$ of size $N \times M$, approximate it as a product of two matrices

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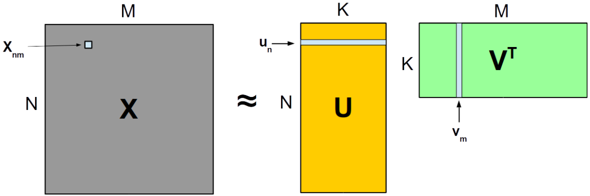

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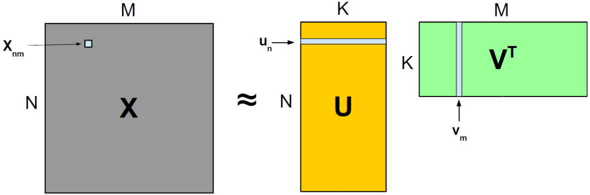

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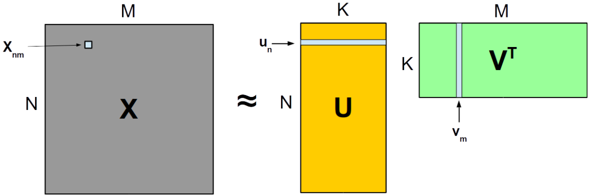

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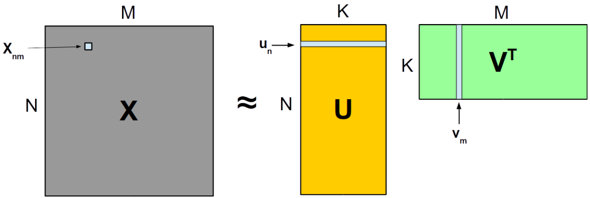

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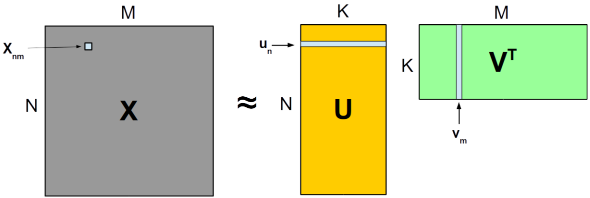

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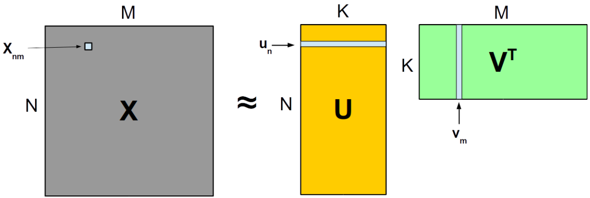

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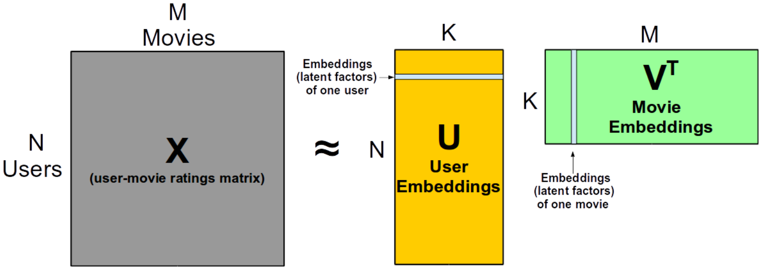
- Each entry of $\mathbf{X}$ can be written as: $X_{nm} \approx \boldsymbol{u}_n^\top \boldsymbol{v}_m = \sum_{k=1}^K u_{nk} v_{mk}$

Why Matrix Factorization?

- The latent factors can be used/interpreted as "embeddings" or "learned features"



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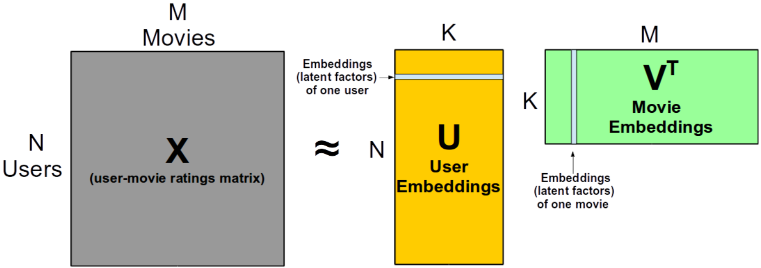

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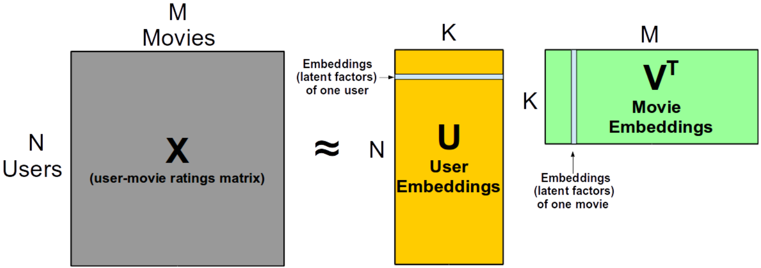

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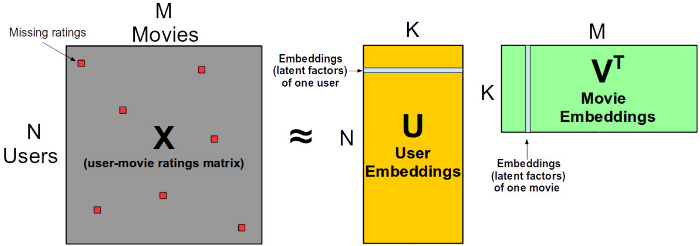
- If $K \ll \min\{M, N\} \Rightarrow$ then can also be seen as dimensionality reduction or a "low-rank factorization" of the matrix $\mathbf{X}$ (somewhat like SVD)

Why Matrix Factorization?

- Can also predict the missing/unknown entries in the original matrix



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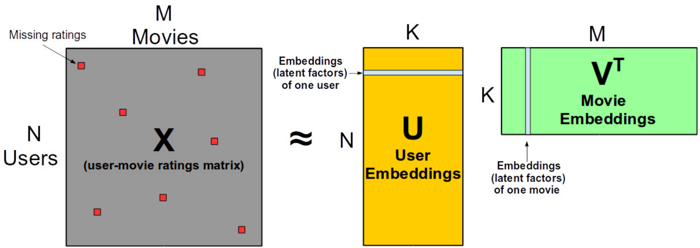

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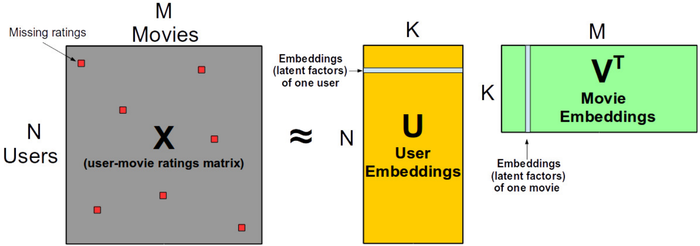

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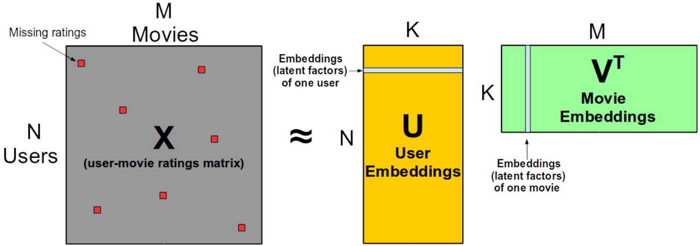

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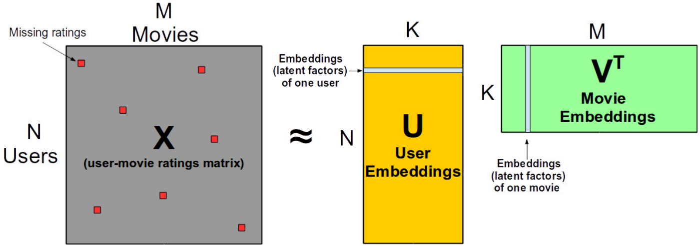

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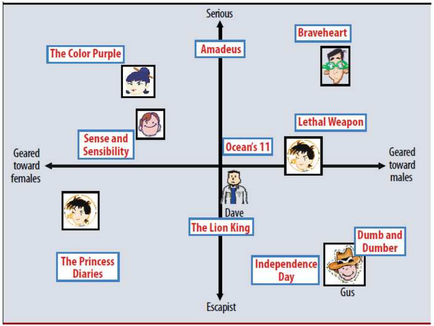
- Note: The "Netflix Challenge" was won by a matrix factorization method

Interpreting the Embeddings/Latent Factors

- Embeddings/latent factors can often be interpreted. E.g., as "genres" if $\mathbf{X}$ represents a user-movie rating matrix. A cartoon with $K = 2$ shown below



Picture courtesy: Matrix Factorization Techniques for Recommender Systems: Koren et al, 2009

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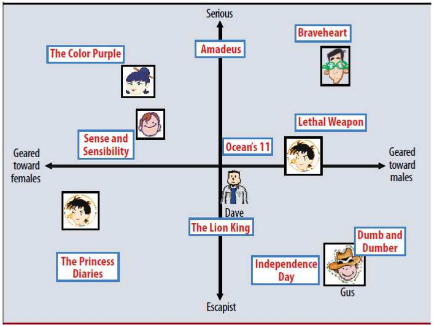

- Similar things (users/movies) get embedded nearby in the embedding space (two things will be deemed similar if their embeddings are similar). Thus useful for computing similarities and/or making recommendations

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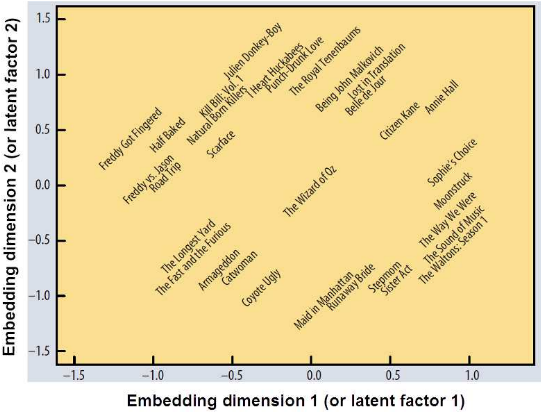
- Another illustation of 2-D embeddings of the movies only



- Similar movies will be embedded at nearby locations in the embedding space

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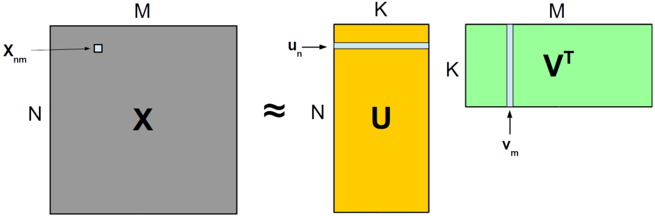
Solving Matrix Factorization

Matrix Factorization

- Recall our matrix factorization model: $\mathbf{X} \approx \mathbf{U}\mathbf{V}^\top$
- Goal: learn $\mathbf{U}$ and $\mathbf{V}$, given a subset Ω of entries in $\mathbf{X}$ (let's call it $\mathbf{X}_\Omega$)
- Recall our notations:
  - $\Omega = \{(n, m)\}$: X_{nm} is observed
  - Ω_{r_n}: column indices of observed entries in row n of $\mathbf{X}$
  - Ω_{c_m}: row indices of observed entries in column m of $\mathbf{X}$



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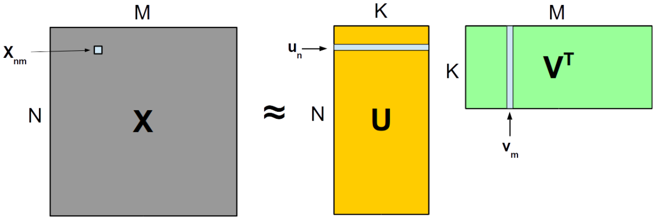
- We want $\mathbf{X}$ to be as close to $\mathbf{U}\mathbf{V}^\top$ as possible



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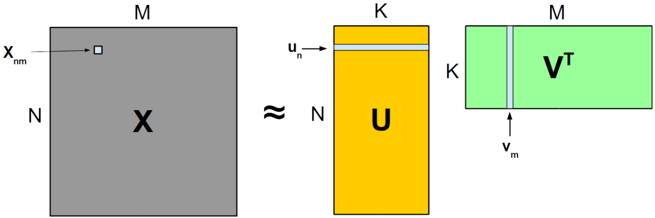

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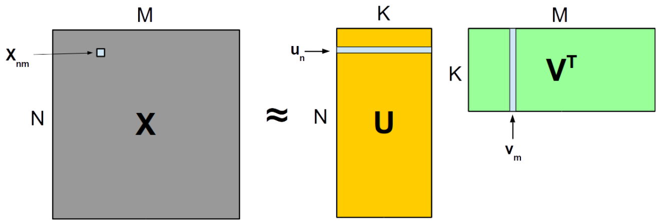

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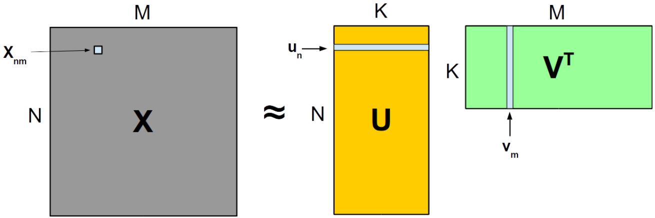

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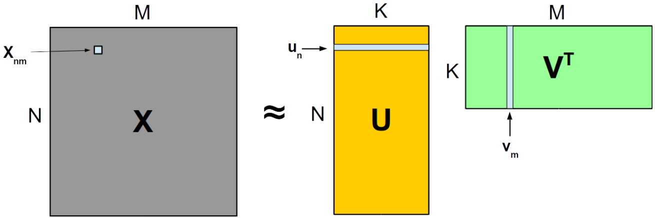

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- Squared loss chosen only for simplicity; other loss functions can be used
- How do we learn $\{\boldsymbol{u}_n\}_{n=1}^N$ and $\{\boldsymbol{v}_m\}_{m=1}^M$?

Alternating Optimization

- We will use an ℓ_2 regularized version of the squared loss function

$$\mathcal{L} = \sum_{(n,m)\in\Omega} (X_{nm} - \boldsymbol{u}_n^\top \boldsymbol{v}_m)^2 + \sum_{n=1}^{N} \lambda_U ||\boldsymbol{u}_n||^2 + \sum_{m=1}^{M} \lambda_V ||\boldsymbol{v}_m||^2$$

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  - $\forall n$, fix all variables except $\boldsymbol{u}_n$ and solve the optim. problem w.r.t. $\boldsymbol{u}_n$

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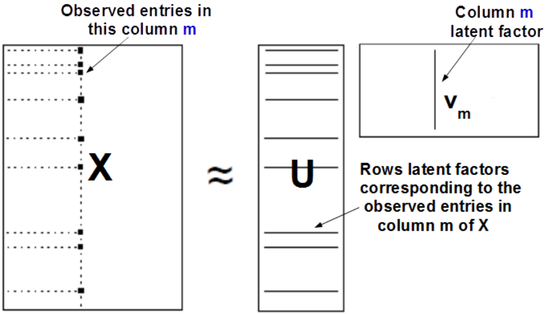
- Thus matrix factorization can be also seen as a sequence of regression problems (one for each latent factor)

Matrix Factorization as Regression

Suppose we are solving for $\boldsymbol{v}_m$ (with $\mathbf{U}$ and all other $\boldsymbol{v}_m$'s fixed)



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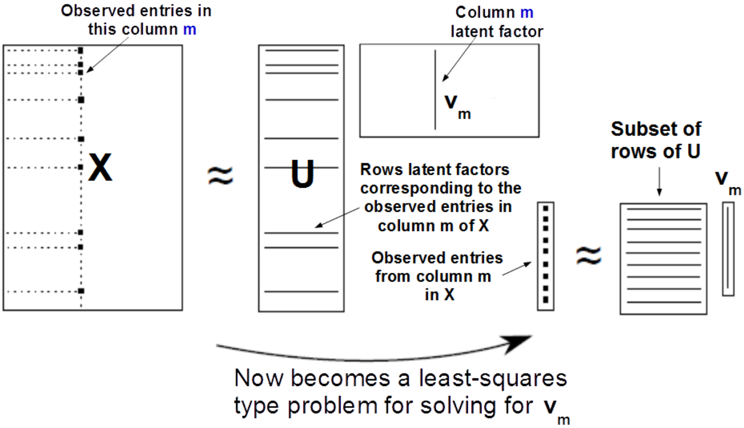

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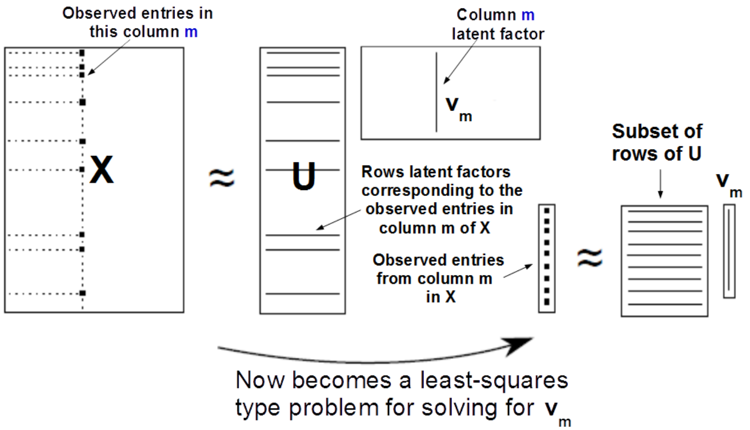

Can think of solving for $\boldsymbol{u}_n$ (with $\mathbf{V}$ and all other $\boldsymbol{u}_n$'s fixed) in the same way

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  - Can think of this also as a probabilistic model (a likelihood function on X_{nm} and priors on the latent factors $\boldsymbol{u}_n$, $\boldsymbol{v}_m$) and do MLE/MAP

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- Final prediction for any (missing) entry: $X_{nm} = \boldsymbol{u}_n^\top \boldsymbol{v}_m$

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- The SGD algorithm chooses a random entry X_{nm} in each iteration, updates $\boldsymbol{u}_n, \boldsymbol{v}_m$, and repeats until convergece ($\boldsymbol{u}_n$'s,$\boldsymbol{v}_m$'s randomly initialized).

Explicit Feedback vs Implicit Feedback Data

- Often the user-item matrix $\mathbf{X}$ is a binary matrix
- $X_{nm} = 1$ means user n watched and liked on the item m

0	0	1	0	1	0
0	0	0	1	0	0
0	1	0	0	0	1
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- Such data needs more careful modeling (other loss functions, not squared/logistic)

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- Such data needs more careful modeling (other loss functions, not squared/logistic)
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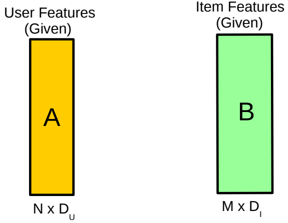

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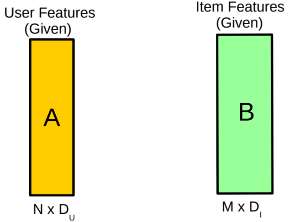

- One possibility now: Use these features/meta-data to get the latent factors for users/items

Matrix Factorization for Inductive Matrix Completion

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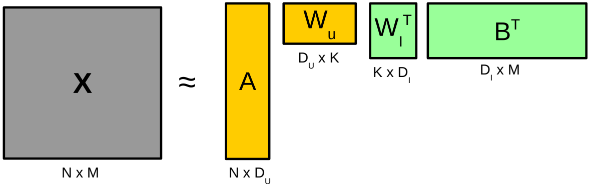

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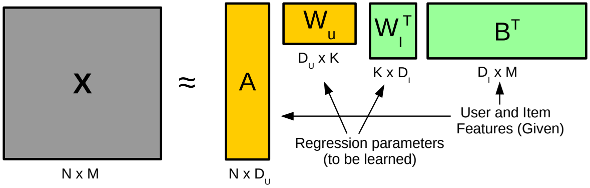

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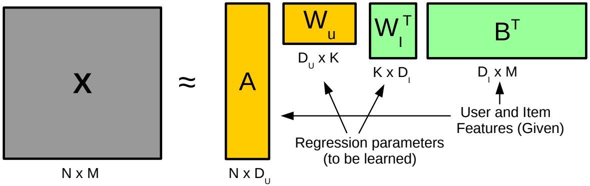

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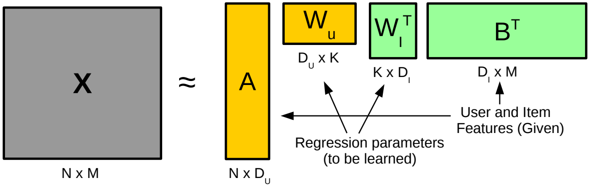

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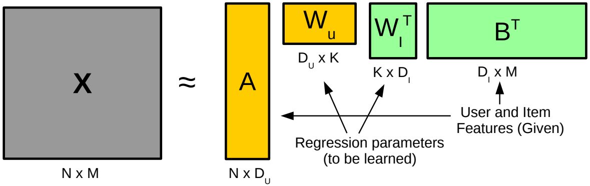

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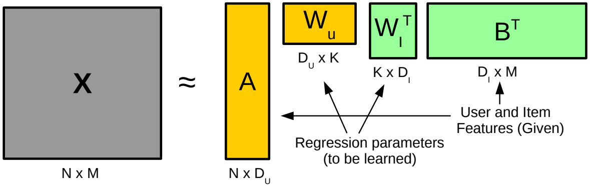

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Some Other Extensions of Matrix Factorization

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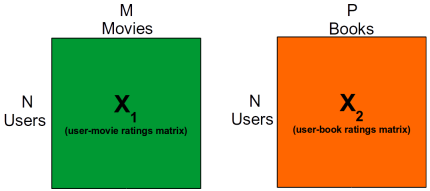

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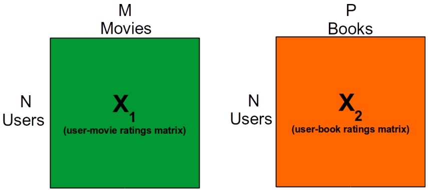

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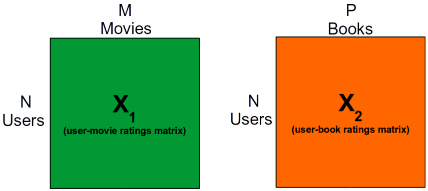

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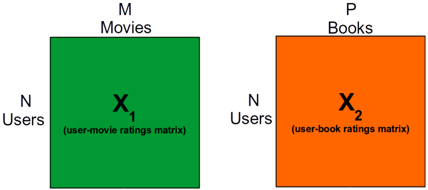

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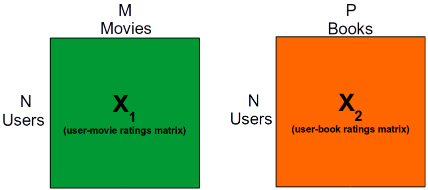

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Tensor Factorization

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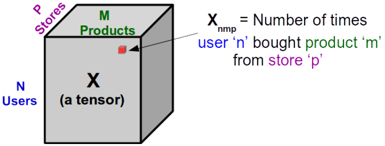
- Consider a 3-dim (or 3-mode or 3-way) tensor $\mathbf{X}$ of size $N \times M \times P$



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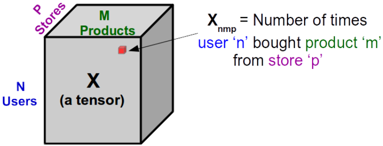

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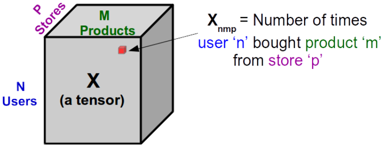

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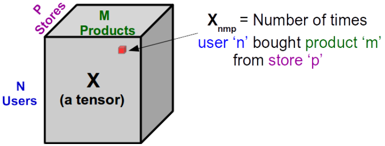

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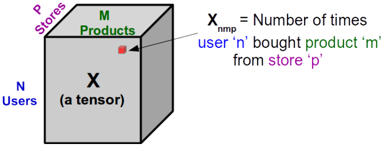

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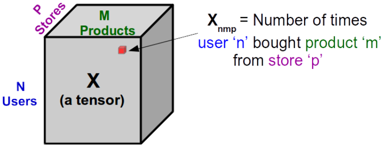

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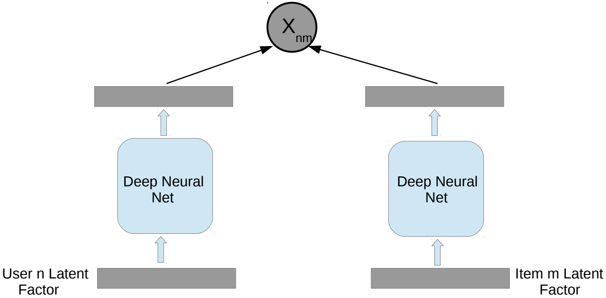
- Several specialized algorithms for tensor factorization (CP/Tucker decomposition, etc.)

(And of course..) Deep Learning based Recommender Systems

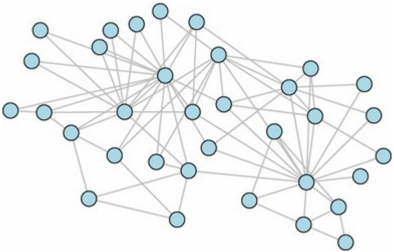
Basic idea: Matrix entries are nonlinear transformations of the latent factors $X_{nm} \approx f(\boldsymbol{u}_n)^\top f(\boldsymbol{v}_m)$

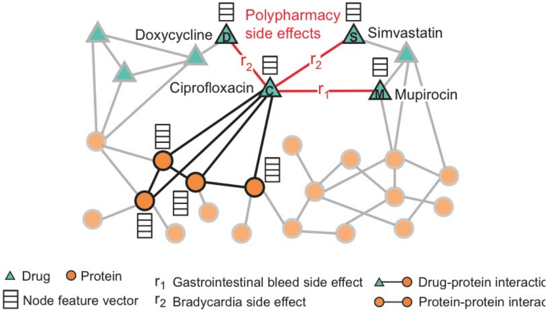


This above is a simple version. Many more sophisticated variants exist (posted a reference on the course webpage in case you are interested in deep learning methods for recommender systems)

Applications to Link Prediction in Graphs

- The user-item matrix is like a bipartite graph
- The matrix factorization ideas we saw today can also be used for any type of graph





- Thus we can get node embeddings as well as a way to do link prediction in such graphs

Right side picture courtesy: snap.standford.edu

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- Still an ongoing area of active research