Dimensionality Reduction (Contd.)

Piyush Rai

Introduction to Machine Learning (CS771A)

October 9, 2018

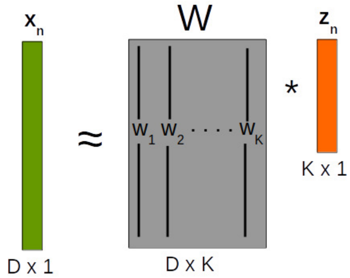
Announcements

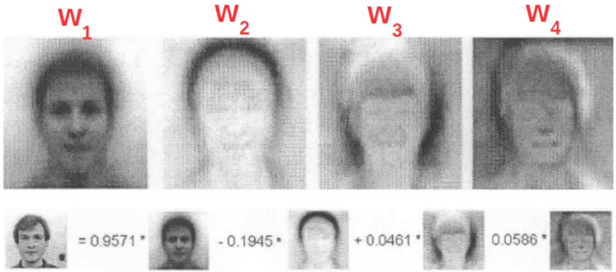
- Quiz graded and scores sent
- Homework 3 out. Due on Oct 31, 11:59pm. Please start early.
- We will finish homework 1 and 2 grading soon
- Start thinking about your course project (if not working on it already)

Recap: Dimensionality Reduction - The Compression View

$$\mathbf{x}_n \approx \sum_{k=1}^{K} z_{nk}\mathbf{w}_k$$

$$\mathbf{x}_n \approx \mathbf{W}\mathbf{z}_n$$

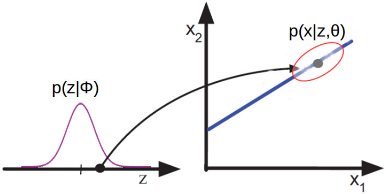




Recap: Probabilistic PCA

A probabilistic model that maps a low-dim $\boldsymbol{z}$ via a linear mapping to generate a high-dim $\boldsymbol{x}$

$$\begin{aligned} \boldsymbol{z}_n &\sim \mathcal{N}(0, \mathbf{I}_K) \\ \boldsymbol{x}_n|\boldsymbol{z}_n &\sim \mathcal{N}(\mathbf{W}\boldsymbol{z}_n, \sigma^2\mathbf{I}_D) \end{aligned}$$



$$p(\boldsymbol{x}_n) = \underbrace{\mathcal{N}(0, \mathbf{W}\mathbf{W}^\top + \sigma^2\mathbf{I}_D)}_{\text{Low-rank Gaussian as } \sigma^2 \to 0}$$

Many improvements possible (non-Gaussian distributions, nonlinear mappings, etc)

Recap: Such Models Can Learn to Generate Real-Looking Data..

- Learn the model parameters from training data $\{\mathbf{x}_1, \ldots, \mathbf{x}_N\}$, e.g., using MLE
- Generate a random $\mathbf{z}$ from $p(\mathbf{z})$ and a random new sample $\mathbf{x}$ conditioned on that $\mathbf{z}$ using $p(\mathbf{x}|\mathbf{z})$

(a) Training data

(b) Random samples

Learning the PPCA Model

- One way: Maximize (conditional) log-likelihood $\sum_{n=1}^{N} \log p(\boldsymbol{x}_n|\boldsymbol{z}_n)$, or minimize its negative

$$\mathcal{L}(\mathbf{Z}, \mathbf{W}, \sigma^2) = \frac{1}{2\sigma^2}\sum_{n=1}^{N} ||\boldsymbol{x}_n - \mathbf{W}\boldsymbol{z}_n||^2 + \frac{ND}{2}\log(2\pi\sigma^2)$$
$$= \frac{1}{2\sigma^2}||\mathbf{X} - \mathbf{Z}\mathbf{W}^\top||_F^2 + \frac{ND}{2}\log(2\pi\sigma^2)$$

- For known σ^2, learning PPCA boils down to solve

$$\{\hat{\mathbf{Z}}, \hat{\mathbf{W}}\} = \arg\min_{\mathbf{Z},\mathbf{W}} ||\mathbf{X} - \mathbf{Z}\mathbf{W}^\top||_F^2$$

- Similar to doing matrix factorization of $\mathbf{X}$ by minimizing the reconstruction error
- Can solve it using ALT-OPT ($\mathbf{Z}$ given $\mathbf{W}$, and $\mathbf{W}$ given $\mathbf{Z}$)
- Another (better) way: will be to do a proper MLE on $\log p(\boldsymbol{x}_n)$

MLE for PPCA

- Doing MLE for PPCA requires maximizing

$$\log p(\mathbf{X}|\Theta) = -\frac{N}{2}(D \log 2\pi + \log |\mathbf{C}| + \text{trace}(\mathbf{C}^{-1}\mathbf{S}))$$

  where $\mathbf{S}$ is the data covariance matrix, $\mathbf{C}^{-1} = \sigma^{-1}\mathbf{I} - \sigma^{-1}\mathbf{W}\mathbf{M}^{-1}\mathbf{W}^\top$and $\mathbf{M} = \mathbf{W}^\top\mathbf{W} + \sigma^2\mathbf{I}$

- Assuming both $\mathbf{W}$ and σ^2 as unknowns, their MLE solution is given by

$$\begin{aligned} \mathbf{W}_{ML} &= \mathbf{U}_K(\mathbf{L}_K - \sigma^2_{ML}\mathbf{I})^{1/2}\mathbf{R} \\ \sigma^2_{ML} &= \frac{1}{D-K}\sum_{k=K+1}^{D} \lambda_k \end{aligned}$$

  where $\mathbf{U}_K$ is $D \times K$ matrix of top K eigvecs of $\mathbf{S}$, $\mathbf{L}_K$: $K \times K$ diagonal matrix of top K eigvals $\lambda_1, \ldots, \lambda_K$, $\mathbf{R}$ is a $K \times K$ arbitrary rotation matrix (equivalent to PCA for $\mathbf{R} = \mathbf{I}$ and $\sigma^2 \to 0$)

- Need to do eigen-decomposition of $D \times D$ data covariance matrix $\mathbf{S}$. EXPENSIVE!!!
- Also, can't do MLE like this if each $\boldsymbol{x}_n$ has some missing entries

† Probabilistic Principal Component Analysis (Tipping and Bishop, 1999)

MLE for PPCA using EM

- Using EM for MLE for PPCA has several benefits
  - No need to do expensive eigen-decomposition
  - Works even when $\boldsymbol{x}_n$ may have some missing entries (HW3 has a problem related to this)
- EM does MLE by maximizing the expected CLL

$$\{\mathbf{W}, \sigma^2\} = \arg\max_{\mathbf{W},\sigma^2} \mathbb{E}_{p(\mathbf{Z}|\mathbf{X},\mathbf{W},\sigma^2)}[\log p(\mathbf{X}, \mathbf{Z}|\mathbf{W}, \sigma^2)]$$

- This is done by iterating between the following two steps
  - E Step: For $n = 1, \ldots, N$, infer the posterior $p(\boldsymbol{z}_n|\boldsymbol{x}_n)$ given current estimate of $\Theta = (\mathbf{W}, \sigma^2)$

$$p(\boldsymbol{z}_n|\boldsymbol{x}_n, \mathbf{W}, \sigma^2) = \mathcal{N}(\mathbf{M}^{-1}\mathbf{W}^\top \boldsymbol{x}_n, \sigma^2\mathbf{M}^{-1}) \qquad (\text{where } \mathbf{M} = \mathbf{W}^\top\mathbf{W} + \sigma^2\mathbf{I}_K)$$

  - M Step: Maximize the expected CLL $\mathbb{E}[p(\mathbf{X}, \mathbf{Z}|\mathbf{W}, \sigma^2)]$ w.r.t. $\mathbf{W}, \sigma^2$

MLE for PPCA using EM

- The expected complete data log-likelihood $\mathbb{E}[\log p(\mathbf{X}, \mathbf{Z}|\mathbf{W}, \sigma^2)]$

$$= -\sum_{n=1}^{N} \left\{ \frac{D}{2} \log \sigma^2 + \frac{1}{2\sigma^2}||\boldsymbol{x}_n||^2 - \frac{1}{\sigma^2}\mathbb{E}[\boldsymbol{z}_n]^\top \mathbf{W}^\top \boldsymbol{x}_n + \frac{1}{2\sigma^2}\text{tr}(\mathbb{E}[\boldsymbol{z}_n \boldsymbol{z}_n^\top]\mathbf{W}^\top \mathbf{W}) + \frac{1}{2}\text{tr}(\mathbb{E}[\boldsymbol{z}_n \boldsymbol{z}_n^\top]) \right\}$$

- Taking the derivative of $\mathbb{E}[\log p(\mathbf{X}, \mathbf{Z}|\mathbf{W}, \sigma^2)]$ w.r.t. $\mathbf{W}$ and setting to zero

$$\mathbf{W} = \left[\sum_{n=1}^{N} \boldsymbol{x}_n \mathbb{E}[\boldsymbol{z}_n]^\top\right] \left[\sum_{n=1}^{N} \mathbb{E}[\boldsymbol{z}_n \boldsymbol{z}_n^\top]\right]^{-1}$$

- To compute $\mathbf{W}$, we need two posterior expectations $\mathbb{E}[\boldsymbol{z}_n]$ and $\mathbb{E}[\boldsymbol{z}_n \boldsymbol{z}_n^\top]$
- These can be easily obtained from the posterior $p(\boldsymbol{z}_n|\boldsymbol{x}_n)$ computed in E step

$$\begin{aligned} p(\boldsymbol{z}_n|\boldsymbol{x}_n, \mathbf{W}) &= \mathcal{N}(\mathbf{M}^{-1}\mathbf{W}^\top \boldsymbol{x}_n, \sigma^2 \mathbf{M}^{-1}) \qquad \text{where } \mathbf{M} = \mathbf{W}^\top \mathbf{W} + \sigma^2 \mathbf{I}_K \\ \mathbb{E}[\boldsymbol{z}_n] &= \mathbf{M}^{-1}\mathbf{W}^\top \boldsymbol{x}_n \\ \mathbb{E}[\boldsymbol{z}_n \boldsymbol{z}_n^\top] &= \mathbb{E}[\boldsymbol{z}_n]\mathbb{E}[\boldsymbol{z}_n]^\top + \text{cov}(\boldsymbol{z}_n) = \mathbb{E}[\boldsymbol{z}_n]\mathbb{E}[\boldsymbol{z}_n]^\top + \sigma^2 \mathbf{M}^{-1} \end{aligned}$$

- Note: The noise variance σ^2 can also be estimated (take deriv., set to zero..)

The Full EM Algorithm for PPCA

- Specify K, initialize $\mathbf{W}$ and σ^2 randomly. Also center the data ($\boldsymbol{x}_n = \boldsymbol{x}_n - \frac{1}{N}\sum_{n=1}^{N} \boldsymbol{x}_n$)
- **E step:** For each n, compute $p(\boldsymbol{z}_n|\boldsymbol{x}_n)$ using current $\mathbf{W}$ and σ^2. Compute exp. for the M step

$$
\begin{aligned}
p(\boldsymbol{z}_n|\boldsymbol{x}_n, \mathbf{W}) &= \mathcal{N}(\mathbf{M}^{-1}\mathbf{W}^\top \boldsymbol{x}_n, \sigma^2\mathbf{M}^{-1}) \qquad \text{where } \mathbf{M} = \mathbf{W}^\top\mathbf{W} + \sigma^2\mathbf{I}_K \\
\mathbb{E}[\boldsymbol{z}_n] &= \mathbf{M}^{-1}\mathbf{W}^\top \boldsymbol{x}_n \\
\mathbb{E}[\boldsymbol{z}_n\boldsymbol{z}_n^\top] &= \text{cov}(\boldsymbol{z}_n) + \mathbb{E}[\boldsymbol{z}_n]\mathbb{E}[\boldsymbol{z}_n]^\top = \mathbb{E}[\boldsymbol{z}_n]\mathbb{E}[\boldsymbol{z}_n]^\top + \sigma^2\mathbf{M}^{-1}
\end{aligned}
$$

- **M step:** Re-estimate $\mathbf{W}$ and σ^2

$$
\begin{aligned}
\mathbf{W}_{new} &= \left[\sum_{n=1}^{N} \boldsymbol{x}_n \mathbb{E}[\boldsymbol{z}_n]^\top\right]\left[\sum_{n=1}^{N} \mathbb{E}[\boldsymbol{z}_n\boldsymbol{z}_n^\top]\right]^{-1} \\
\sigma^2_{new} &= \frac{1}{ND}\sum_{n=1}^{N}\left\{||\boldsymbol{x}_n||^2 - 2\mathbb{E}[\boldsymbol{z}_n]^\top\mathbf{W}_{new}^\top\boldsymbol{x}_n + \text{tr}\left(\mathbb{E}[\boldsymbol{z}_n\boldsymbol{z}_n^\top]\mathbf{W}_{new}^\top\mathbf{W}_{new}\right)\right\}
\end{aligned}
$$

- Set $\mathbf{W} = \mathbf{W}_{new}$ and $\sigma^2 = \sigma^2_{new}$. If not converged (monitor $p(\mathbf{X}|\Theta)$), go back to E step
- **Note:** For $\sigma^2 = 0$, this EM algorithm can also be used to efficiently solve standard PCA (note that this EM algorithm doesn't require any eigen-decomposition)
- Missing entries of $\boldsymbol{x}_n$ can be estimated in the E step as $p(\boldsymbol{x}_n^{miss}|\boldsymbol{x}_n^{obs})$

Why center the data before doing PPCA?

- The PPCA model, for each $\boldsymbol{x}_n, n = 1, \ldots, N$, can also be writtten as

$$\boldsymbol{x}_n = \mu + \mathbf{W}\boldsymbol{z}_n + \epsilon_n \qquad \text{where} \quad \epsilon_n \sim \mathcal{N}(0, \sigma^2 \mathbf{I}_D)$$

- The marginal distribution is

$$p(\boldsymbol{x}_n) = \mathcal{N}(\mu, \mathbf{W}\mathbf{W}^\top + \sigma^2 \mathbf{I}_D)$$

- The MLE of μ is simply $\frac{1}{N}\sum_{n=1}^{N} \boldsymbol{x}_n$
- So we can simply subtract μ from each observation and assume

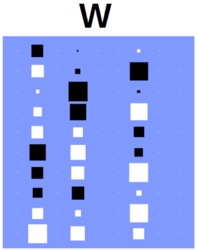
$$\boldsymbol{x}_n = \mathbf{W}\boldsymbol{z}_n + \epsilon_n$$

... and apply PPCA without μ

How to Set "K"?

- Several option to select the "best" K, e.g.,
  - Look at AIC/BIC criteria (NLL + KD or NLL + $K \log D$) and pick the one with smallest K
  - Use sparsity inducing priors on $\mathbf{W}$ and/or $\boldsymbol{z}_n$ (set K to some large value; the unnecessary columns of $\mathbf{W}$ will "turn off" automatically as they will be shrunk to zero during inference)

Using sparsity-inducing Prior (e.g., Automatic Relevance Determination) on **W**

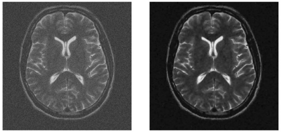


Effect: Only few columns of **W** will have entries with significant magnitudes

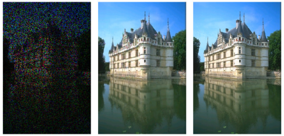
  - Compute the marginal likelihood (or its approximation) for each K and choose the best model
  - Nonparametric Bayesian methods (allow K to grow with data)

Some Applications of PCA/PPCA

- Compression/dimensionality reduction is a natural application (use $\boldsymbol{z}_n$ instead of $\boldsymbol{x}_n$)
- Also used for learning low-dim. "good" features $\boldsymbol{z}_n$ from high-dim noisy features $\boldsymbol{x}_n$
  - Note that this is different from feature selection ($\boldsymbol{z}_n$ is a transformed version of $\boldsymbol{x}_n$, not a subset)
- Learning the noise variance enables "image denoising": $\boldsymbol{x}_n = \mathbf{W}\boldsymbol{z}_n + \epsilon_n$; $\mathbf{W}\boldsymbol{z}_n$ is the "clean" part

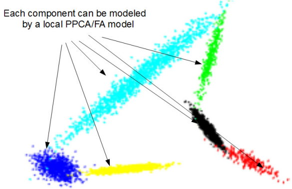


- Ability to fill-in missing data enables "image inpainting" (left: image with 80% missing data, middle: reconstructed, right: original)



Mixture of PPCA

- May be appropriate if data also exists in clusters (suppose $M > 1$ clusters)
- Data in each cluster (say m) can have its own "local" PPCA model defined by $\{\mu_m, \mathbf{W}_m, \sigma_m^2\}$
- Can use M such PPCA models $\{\mu_m, \mathbf{W}_m, \sigma_m^2\}_{m=1}^{M}$ (one per cluster) for the entire data



- Mixtures of PPCA can be seen as playing several roles
    - Jointly learning clustering and dimensionality reduction
    - Nonlinear dimensionality reduction
    - A flexible probability density model: Mixture of low-rank Gaussians

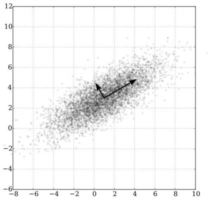
Mixture of PPCA

- For mixture of PPCA, the generative story for each observation $\boldsymbol{x}_n$ is as follows
  - Generate its cluster id as
  $$\boldsymbol{c}_n \sim \text{multinoulli}(\pi_1, \ldots, \pi_M)$$
  - Generate latent variable $\boldsymbol{z}_n \in \mathbb{R}^K$ as
  $$\boldsymbol{z}_n \sim \mathcal{N}(\mathbf{0}, \mathbf{I}_K)$$
  - Generate obervation $\boldsymbol{x}_n \in \mathbb{R}^D$ from the $\boldsymbol{c}_n^{th}$ PPCA/FA model
  $$\boldsymbol{x}_n \sim \mathcal{N}(\boldsymbol{\mu}_{\boldsymbol{c}_n} + \mathbf{W}_{\boldsymbol{c}_n} \boldsymbol{z}_n, \sigma^2_{\boldsymbol{c}_n} \mathbf{I}_D)$$
- Each PPCA model has its separate mean $\boldsymbol{\mu}_{\boldsymbol{c}_n}$ (not needed when $M = 1$ if data is centered)
- **Exercise:** What will be the marginal distribution of $\boldsymbol{x}_n$, i.e.. $p(\boldsymbol{x}_n | \Theta)$?
- **Exercise:** Use EM in this model to learn the parameters and latent variables

(Classic) Principal Component Analysis

Principal Component Analysis (PCA)

- A classic linear dim. reduction method (Pearson, 1901; Hotelling, 1930)
- Can be seen as
  - Learning projection directions that capture maximum variance in data
  - Learning projection directions that result in smallest reconstruction error
- Can also be seen as changing the basis in which the data is represented (and transforming the features such that new features become decorrelated)

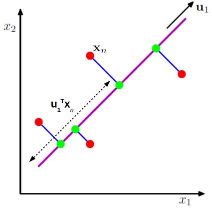


- Also related to other classic methods, e.g., Factor Analysis (Spearman, 1904)

PCA as Maximizing Variance

Variance Captured by Projections

- Consider projecting $\boldsymbol{x}_n \in \mathbb{R}^D$ on a one-dim subspace (basically, a line) defined by $\boldsymbol{u}_1 \in \mathbb{R}^D$
- Projection/embedding of $\boldsymbol{x}_n$ along a one-dim subspace $\boldsymbol{u}_1 = \boldsymbol{u}_1^\top \boldsymbol{x}_n$ (location of the green point along the purple line representing $\boldsymbol{u}_1$)

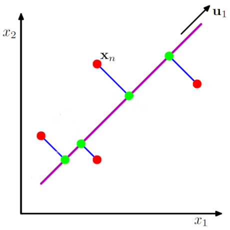


- Mean of projections of all the data: $\frac{1}{N}\sum_{n=1}^N \boldsymbol{u}_1^\top \boldsymbol{x}_n = \boldsymbol{u}_1^\top(\frac{1}{N}\sum_{n=1}^N \boldsymbol{x}_n) = \boldsymbol{u}_1^\top \boldsymbol{\mu}$
- Variance of the projected data ("spread" of the green points)

$$\frac{1}{N}\sum_{n=1}^N \left(\boldsymbol{u}_1^\top \boldsymbol{x}_n - \boldsymbol{u}_1^\top \boldsymbol{\mu}\right)^2 = \frac{1}{N}\sum_{n=1}^N \{\boldsymbol{u}_1^\top(\boldsymbol{x}_n - \boldsymbol{\mu})\}^2 = \boldsymbol{u}_1^\top \mathbf{S} \boldsymbol{u}_1$$

- **S** is the $D \times D$ **data covariance matrix**: $\boldsymbol{S} = \frac{1}{N}\sum_{n=1}^N (\boldsymbol{x}_n - \boldsymbol{\mu})(\boldsymbol{x}_n - \boldsymbol{\mu})^\top$. If data already centered ($\boldsymbol{\mu} = 0$) then $\boldsymbol{S} = \frac{1}{N}\sum_{n=1}^N \boldsymbol{x}_n \boldsymbol{x}_n^\top = \frac{1}{N}\mathbf{X}^\top\mathbf{X}$

Direction of Maximum Variance



- We want $\boldsymbol{u}_1$ s.t. the variance of the projected data is maximized

$$\arg\max_{\boldsymbol{u}_1} \; \boldsymbol{u}_1^\top \mathbf{S} \boldsymbol{u}_1$$

- To prevent trivial solution (max var. = infinite), assume $||\boldsymbol{u}_1|| = 1 = \boldsymbol{u}_1^\top \boldsymbol{u}_1$
- We will find $\boldsymbol{u}_1$ by solving the following constrained opt. problem

$$\boxed{\arg\max_{\boldsymbol{u}_1} \; \boldsymbol{u}_1^\top \mathbf{S} \boldsymbol{u}_1 + \lambda_1(1 - \boldsymbol{u}_1^\top \boldsymbol{u}_1)}$$

where λ_1 is a Lagrange multiplier

Direction of Maximum Variance

- The objective function: $\arg\max_{\boldsymbol{u}_1} \ \boldsymbol{u}_1^\top \mathbf{S} \boldsymbol{u}_1 + \lambda_1(1 - \boldsymbol{u}_1^\top \boldsymbol{u}_1)$
- Taking the derivative w.r.t. $\boldsymbol{u}_1$ and setting to zero gives

$$\mathbf{S}\boldsymbol{u}_1 = \lambda_1 \boldsymbol{u}_1$$

- Thus $\boldsymbol{u}_1$ is an eigenvector of $\mathbf{S}$ (with corresponding eigenvalue λ_1)
- But which of $\mathbf{S}$'s (D possible) eigenvectors it is?
- Note that since $\boldsymbol{u}_1^\top \boldsymbol{u}_1 = 1$, the variance of projected data is

$$\boldsymbol{u}_1^\top \mathbf{S} \boldsymbol{u}_1 = \lambda_1$$

- Var. is maximized when $\boldsymbol{u}_1$ is the (top) eigenvector with largest eigenvalue
- The top eigenvector $\boldsymbol{u}_1$ is also known as the first Principal Component (PC)
- Other directions can also be found likewise (with each being orthogonal to all previous ones) using the eigendecomposition of $\mathbf{S}$ (this is PCA)

Principal Component Analysis

- Center the data (subtract the mean $\boldsymbol{\mu} = \frac{1}{N}\sum_{n=1}^{N} \boldsymbol{x}_n$ from each data point)
- Compute the covariance matrix $\mathbf{S}$ using the centered data as

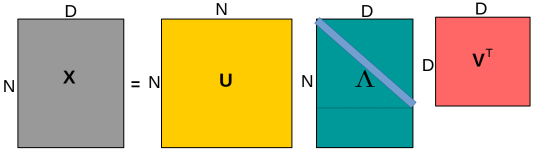
$$\mathbf{S} = \frac{1}{N}\mathbf{X}^\top\mathbf{X}$$

- Do an eigendecomposition of the covariance matrix $\mathbf{S}$
- Take first K leading eigenvectors $\{\boldsymbol{u}_k\}_{k=1}^{K}$ with eigenvalues $\{\lambda_k\}_{k=1}^{K}$
- The final K dim. projection/embedding of data is given by

$$\mathbf{Z} = \mathbf{X}\mathbf{U}$$

where $\mathbf{U} = [\boldsymbol{u}_1 \ \ldots \ \boldsymbol{u}_K]$ is $D \times K$ and embedding matrix $\mathbf{Z}$ is $K \times N$

Singular Value Decomposition (SVD)



- We can represent any matrix $\mathbf{X}$ of size $N \times D$ using SVD as $\mathbf{X} = \mathbf{U}\Lambda\mathbf{V}^\top$
- $\mathbf{U} = [\boldsymbol{u}_1, \ldots, \boldsymbol{u}_N]$ is $N \times N$, each $\boldsymbol{u}_n \in \mathbb{R}^N$ a left singular vector of $\mathbf{X}$
  - $\mathbf{U}$ is orthonormal: $\boldsymbol{u}_n^\top \boldsymbol{u}_{n'} = 0$ for $n \neq n'$, and $\boldsymbol{u}_n^\top \boldsymbol{u}_n = 1 \Rightarrow \mathbf{U}\mathbf{U}^\top = \mathbf{I}_N$
- Λ is $N \times D$ with only $\min(N, D)$ diagonal entries (all positive) - singular values (decreasing order)
- $\mathbf{V} = [\boldsymbol{v}_1, \ldots, \boldsymbol{v}_D]$ is $D \times D$, each $\boldsymbol{v}_d \in \mathbb{R}^D$, a right singular vector of $\mathbf{X}$
  - $\mathbf{V}$ is orthonormal: $\boldsymbol{v}_d^\top \boldsymbol{v}_{d'} = 0$ for $d \neq d'$, and $\boldsymbol{v}_d^\top \boldsymbol{v}_d = 1 \Rightarrow \mathbf{V}\mathbf{V}^\top = \mathbf{I}_D$
- Note: If $\mathbf{X}$ is symmetric then it is known as eigenvalue decomposition (and $\mathbf{U} = \mathbf{V}$ in that case)

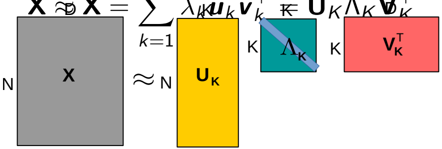
Low-Rank Approximation via SVD

- Can also expand the SVD expression as

$$\mathbf{X} = \sum_{k=1}^{\min(N,D)} \lambda_k \boldsymbol{u}_k \boldsymbol{v}_k^\top$$

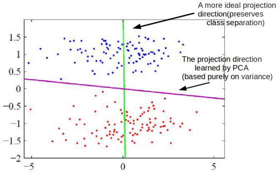
- Can write a rank-K approximation of $\mathbf{X}$ (where $K \ll \min(N, D)$) as

$$\mathbf{X} \approx \hat{\mathbf{X}} = \sum_{k=1}^{K} \lambda_k \boldsymbol{u}_k \boldsymbol{v}_k^\top = \mathbf{U}_K \Lambda_K \mathbf{V}_K^\top$$



PCA/PPCA: Limitations and Extensions

- A linear projection method
  - Won't work well if data can't be approximated by a linear subspace
  - But PCA/PPCA can be kernelized (Kernel PCA or Gaussian Process Latent Variable Models)
- Variance based projection directions can sometimes be suboptimal (e.g., if we want to preserve class separation, e.g., when doing classification)



- PCA relies on eigendecomposition of an $D \times D$ covariance matrix
  - Can be slow if done naïvely. Takes $O(D^3)$ time
  - Many faster methods exists (e.g., Power Method)
  - Note: PPCA doesn't suffer from this issue (EM can be very efficient!)

Next Class

- How to compute singular vectors (SVD) - power method
- Nonlinear Dimensionality Reduction
- Supervised Dimensionality Reduction
- Dimensionality Reduction for Visualization