Parameter Estimation in Latent Variable Models

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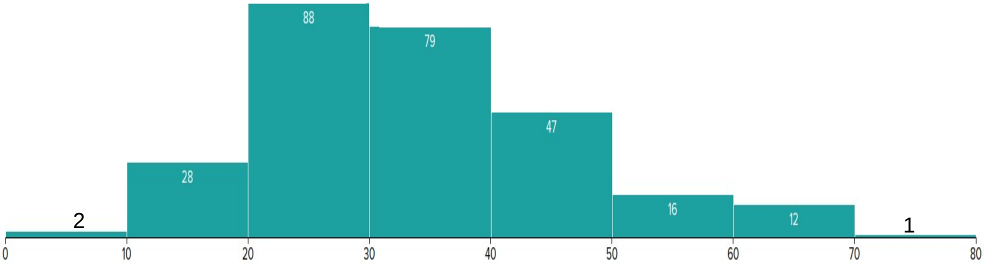
Introduction to Machine Learning (CS771A)

September 25, 2018

Some Mid-Sem Statistics



MINIMUM	MEDIAN	MAXIMUM	MEAN	STD DEV
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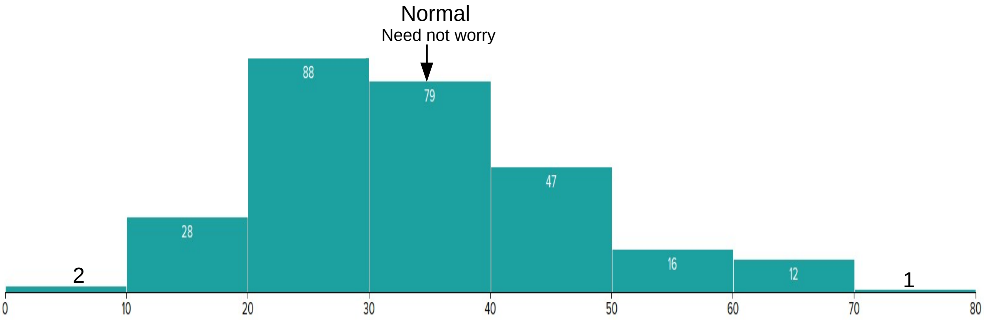

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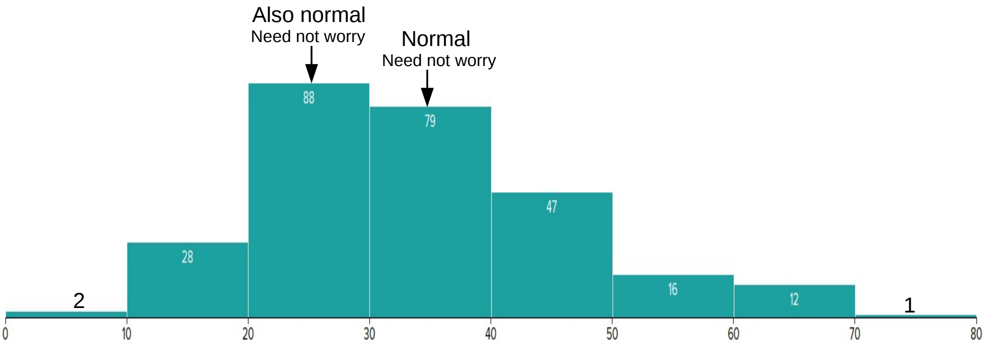

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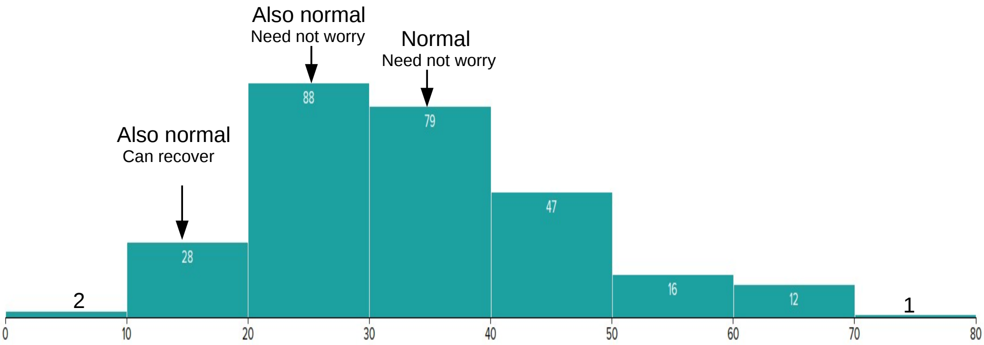

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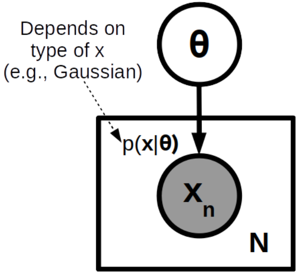
Latent Variable Models

A Simple Generative Model

- All observations $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N\}$ generated from a distribution $p(\boldsymbol{x}|\theta)$



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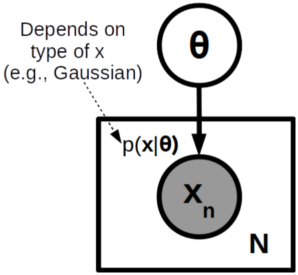

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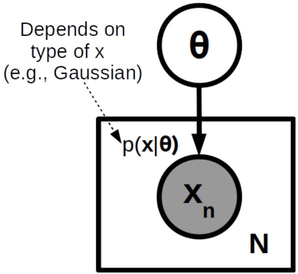

- Unknowns: Parameters θ of the assumed data distribution $p(\boldsymbol{x}|\theta)$
- Many ways to estimate the parameters (MLE, MAP, or Bayesian inference)

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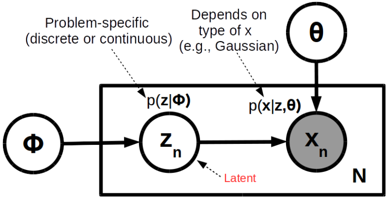

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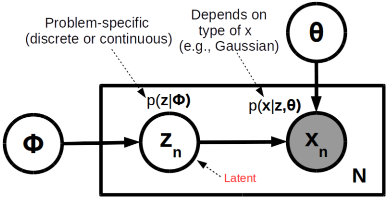

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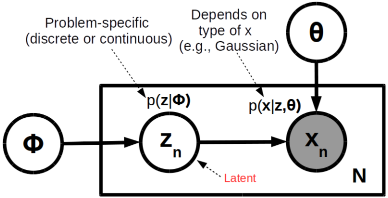

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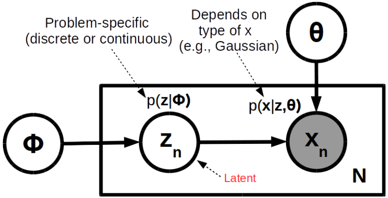

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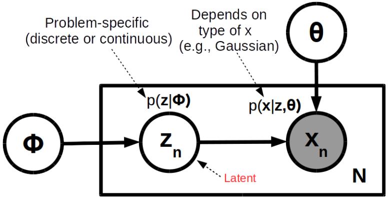

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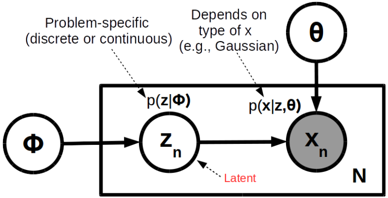

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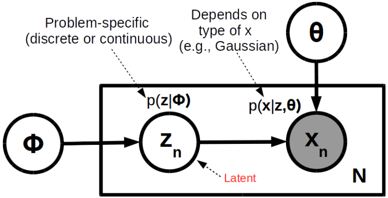

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Brief Detour/Recap:
Gaussian Parameter Estimation

MLE for Multivariate Gaussian

- Multivariate Gaussian in D dimensions

$$p(\boldsymbol{x}|\mu, \Sigma) = \frac{1}{(2\pi)^{D/2}|\Sigma|^{1/2}} \exp\left(-\frac{1}{2}(\boldsymbol{x} - \mu)^\top \Sigma^{-1}(\boldsymbol{x} - \mu)\right)$$

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- In general, when the distribution is an exponential family distribution, MLE is usually very easy

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Many well-known distribution (Bernoulli, Binomial, multinoulli, beta, gamma, Gaussian, etc.) are exponential family distributions

`https://en.wikipedia.org/wiki/Exponential_family`

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- Basically estimating K Gaussians instead of just 1 (each using data only from that class)

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$$= \arg\max_{\Theta} \prod_{n=1}^{N} \prod_{k=1}^{K} [p(y_n = k|\Theta)p(\boldsymbol{x}_n|y_n = k, \Theta)]^{y_{nk}}$$

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&= \arg\max_{\Theta} \prod_{n=1}^{N} \prod_{k=1}^{K} [p(y_n = k|\Theta)p(\boldsymbol{x}_n|y_n = k, \Theta)]^{y_{nk}} \\
&= \arg\max_{\Theta} \log \prod_{n=1}^{N} \prod_{k=1}^{K} [p(y_n = k|\Theta)p(\boldsymbol{x}_n|y_n = k, \Theta)]^{y_{nk}}
\end{aligned}$$

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MLE for Generative Classification with Gaussian Class-conditionals

- Let's look at the "formal" procedure of deriving MLE in this case
- MLE for $\Theta = \{\pi_k, \mu_k, \Sigma_k\}_{k=1}^{K}$ in this case can be written as (assuming i.i.d. data)

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- Given $(\mathbf{X}, \boldsymbol{y})$, optimizing it w.r.t. π_k, μ_k, Σ_k will give us the solution we saw on the previous slide

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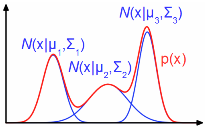

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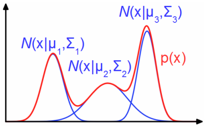

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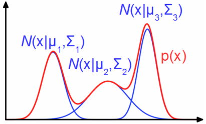

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  - Semi-supervised generative classification: In training data, some y_n's are known, some not known

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- Recall the MLE problem for Θ when the labels are known

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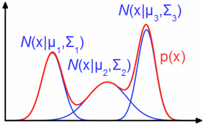

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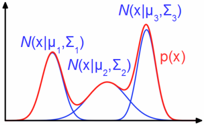

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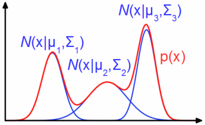

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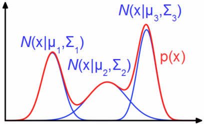

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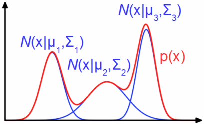

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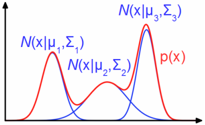

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- One workaround: Can try doing alternating optimization

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- Based on the fact that MLE is simple when labels are known

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3. Given $\hat{\mathbf{Z}} = \{\hat{z}_1, \ldots, \hat{z}_N\}$, re-estimate Θ using MLE

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MLE for Gaussian Mixture Model using ALT-OPT

1. Initialize Θ as $\hat{\Theta}$
2. For $n = 1, \ldots, N$, find the best z_n

$$\begin{aligned}\hat{z}_n &= \arg\max_{k\in\{1,\ldots,K\}} p(\boldsymbol{x}_n, z_n = k|\hat{\Theta}) \\ &= \arg\max_{k\in\{1,\ldots,K\}} p(z_n = k|\boldsymbol{x}_n, \hat{\Theta})\end{aligned}$$

3. Given $\hat{\mathbf{Z}} = \{\hat{z}_1, \ldots, \hat{z}_N\}$, re-estimate Θ using MLE

$$\hat{\Theta} = \arg\max_{\Theta} \sum_{n=1}^{N} \sum_{k=1}^{K} \hat{z}_{nk}[\log \pi_k + \log \mathcal{N}(\boldsymbol{x}_n|\mu_k, \Sigma_k)]$$

4. Go to step 2 if not yet converged

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- It turns out (as we will see), this ALT-OPT is an approximation of the Expectation Maximization (EM) algorithm for GMM

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- Also, if we choose $q(\mathbf{Z}) = p(\mathbf{Z}|\mathbf{X},\Theta)$, then $\log p(\mathbf{X}|\Theta) = \mathbb{E}_{q(\mathbf{Z})}\left[\log \frac{p(\mathbf{X},\mathbf{Z}|\Theta)}{q(\mathbf{Z})}\right]$

EM for GMM

- The EM algorithm for GMM does the following

$$\hat{\Theta}_{new} = \arg\max_{\Theta} \sum_{n=1}^{N} \sum_{k=1}^{K} \mathbb{E}[z_{nk}][\log \pi_k + \log \mathcal{N}(\boldsymbol{x}_n|\mu_k, \Sigma_k)]$$

.. which is nothing but maximizing $\mathbb{E}_{q(\mathbf{Z})}[\log p(\mathbf{X}, \mathbf{Z}|\Theta)]$ with $q(\mathbf{Z}) = p(\mathbf{Z}|\mathbf{X}, \hat{\Theta}_{old})$

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- Next class: Details of EM for GMM, special cases, and the general EM algorithm and its properties