SVM (Contd), Multiclass and One-Class SVM

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Introduction to Machine Learning (CS771A)

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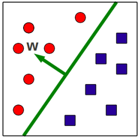
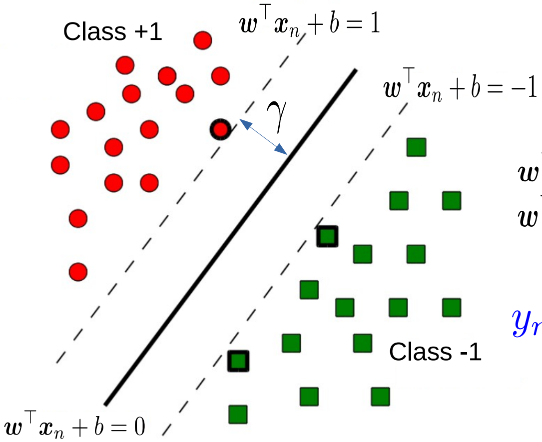
Recap: Hyperplane-based Classification

- Basic idea: Learn to separate by a hyperplane $\boldsymbol{w}^\top \boldsymbol{x} + b = 0$



- Predict the label of a test input $\boldsymbol{x}_*$ as: $\hat{y}_* = \text{sign}(\boldsymbol{w}^\top \boldsymbol{x}_* + b)$
- The hyperplane may be "implied" by the model, or learned directly
  - Implied: Prototype-based classification, nearest neighbors, generative classification, etc.
  - Directly learned: Logistic regression, Perceptron, Support Vector Machine, etc.
- The "direct" approach defines a model with parameters $\boldsymbol{w}$ (and optionally b) and learns them by minimizing a suitable loss function (and doesn't model $\boldsymbol{x}$, i.e., purely discriminative)
- The hyperplane need not be linear (e.g., can be made nonlinear using kernel methods - next class)

Recap: Hyperplanes and Margin



$$\boldsymbol{w}^\top \boldsymbol{x}_n + b \geq 1 \text{ for } y_n = +1$$
$$\boldsymbol{w}^\top \boldsymbol{x}_n + b \leq -1 \text{ for } y_n = -1$$

$$y_n(\boldsymbol{w}^\top \boldsymbol{x}_n + b) \geq 1$$

$$\gamma = \min_{1 \leq n \leq N} \frac{|\boldsymbol{w}^\top \boldsymbol{x}_n + b|}{||\boldsymbol{w}||} = \frac{1}{||\boldsymbol{w}||}$$

(Margin on each side)

Recap: Maximum-Margin Hyperplane

Total margin (2γ)

$$(\hat{\boldsymbol{w}}, \hat{b}) = \arg\max_{\boldsymbol{w}, b} \frac{2}{||\boldsymbol{w}||}, \quad \text{s.t.} \quad y_n(\boldsymbol{w}^\top \boldsymbol{x}_n + b) \geq 1$$

(equivalent to)

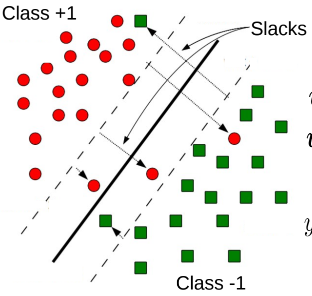
$$(\hat{\boldsymbol{w}}, \hat{b}) = \arg\min_{\boldsymbol{w}, b} \frac{||\boldsymbol{w}||^2}{2}, \quad \text{s.t.} \quad y_n(\boldsymbol{w}^\top \boldsymbol{x}_n + b) \geq 1$$

Hard-margin SVM

("hard" = want all points to satisfy the margin constraint)

Recap: Maximum-Margin Hyperplane with Slacks

- Still want a max-margin hyperplane but want to relax the hard constraint $y_n(\boldsymbol{w}^\top \boldsymbol{x}_n + b) \geq 1$
- Let's allow every point $\boldsymbol{x}_n$ to "slack the constraint" by a distance $\xi_n \geq 0$



$$\boldsymbol{w}^\top \boldsymbol{x}_n + b \geq 1 - \xi_n \text{ for } y_n = +1$$
$$\boldsymbol{w}^\top \boldsymbol{x}_n + b \leq -1 + \xi_n \text{ for } y_n = -1$$

⬇

$$y_n(\boldsymbol{w}^T \boldsymbol{x}_n + b) \geq 1 - \xi_n, \quad \xi_n \geq 0$$

- Points with $\xi_n \geq 0$ will be either in the margin region or totally on the wrong side
- **New Objective:** Maximize the margin while keeping the sum of slacks $\sum_{n=1}^{N} \xi_n$ small
- **Note:** Can also think of the sum of slacks as the total training error

Recap: Maximum-Margin Hyperplane with Slacks

$$(\hat{\boldsymbol{w}}, \hat{b}, \boldsymbol{\xi}) = \arg\min_{\boldsymbol{w}, b} \frac{||\boldsymbol{w}||^2}{2} + C \sum_{n=1}^{N} \xi_n$$

Maximize the margin

Hyperparmeter to balance the two

Minimize the sum of slacks (don't have too many violations)

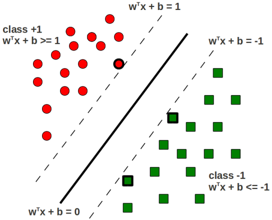
$$\text{s.t.} \quad y_n(\boldsymbol{w}^\top \boldsymbol{x}_n + b) \geq 1 - \xi_n$$

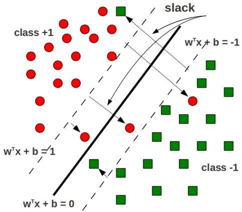
Slack-relaxed constraints

$$\xi_n \geq 0$$

- This formulation is known as the "soft-margin" SVM
- Very small C: Large margin but also large training error. :-(
- Very large C: Small training error but also small margin. :-(
- C controls the trade-off between large margin and small training error

Summary: Hard-Margin SVM vs Soft-Margin SVM





- Objective for the hard-margin SVM (unknowns are $\boldsymbol{w}$ and b)

$$\arg\min_{\boldsymbol{w},b} \frac{||\boldsymbol{w}||^2}{2}$$
$$\text{subject to} \quad y_n(\boldsymbol{w}^T\boldsymbol{x}_n + b) \geq 1, \qquad n = 1,\ldots,N$$

- Objective for the soft-margin SVM (unknowns are $\boldsymbol{w}$, b, and $\{\xi_n\}_{n=1}^N$)

$$\arg\min_{\boldsymbol{w},b,\boldsymbol{\xi}} \frac{||\boldsymbol{w}||^2}{2} + C\sum_{n=1}^{N}\xi_n$$
$$\text{subject to} \quad y_n(\boldsymbol{w}^T\boldsymbol{x}_n + b) \geq 1 - \xi_n, \quad \xi_n \geq 0 \qquad n = 1,\ldots,N$$

- In either case, we have to solve a constrained, convex optimization problem

Solving SVM Objectives

Solving Hard-Margin SVM

- The hard-margin SVM optimization problem is:

$$\begin{aligned} &\arg\min_{\boldsymbol{w},b} \frac{||\boldsymbol{w}||^2}{2} \\ &\text{subject to} \quad 1 - y_n(\boldsymbol{w}^T\boldsymbol{x}_n + b) \le 0, \qquad n = 1, \ldots, N \end{aligned}$$

- A constrained optimization problem. Can solve using Lagrange's method
- Introduce Lagrange Multipliers α_n $(n = \{1, \ldots, N\})$, one for each constraint, and solve

$$\min_{\boldsymbol{w},b} \max_{\boldsymbol{\alpha} \ge 0} \mathcal{L}(\boldsymbol{w}, b, \boldsymbol{\alpha}) = \frac{||\boldsymbol{w}||^2}{2} + \sum_{n=1}^{N} \alpha_n \{1 - y_n(\boldsymbol{w}^T\boldsymbol{x}_n + b)\}$$

- Note: $\boldsymbol{\alpha} = [\alpha_1, \ldots, \alpha_N]$ is the vector of Lagrange multipliers
- Note: It is easier (and helpful; we will soon see why) to solve the dual problem: min and then max

Solving Hard-Margin SVM

- The dual problem (min then max) is

$$\max_{\boldsymbol{\alpha}\geq 0} \min_{\boldsymbol{w},b} \mathcal{L}(\boldsymbol{w}, b, \boldsymbol{\alpha}) = \frac{\boldsymbol{w}^\top \boldsymbol{w}}{2} + \sum_{n=1}^{N} \alpha_n \{1 - y_n(\boldsymbol{w}^T \boldsymbol{x}_n + b)\}$$

- Take (partial) derivatives of $\mathcal{L}$ w.r.t. $\boldsymbol{w}$, b and set them to zero

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{w}} = 0 \Rightarrow \boxed{\boldsymbol{w} = \sum_{n=1}^{N} \alpha_n y_n \boldsymbol{x}_n} \qquad \frac{\partial \mathcal{L}}{\partial b} = 0 \Rightarrow \sum_{n=1}^{N} \alpha_n y_n = 0$$

- Important: Note the form of the solution $\boldsymbol{w}$ - it is simply a weighted sum of all the training inputs $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N$ (and α_n is like the "importance" of $\boldsymbol{x}_n$)
- Substituting $\boldsymbol{w} = \sum_{n=1}^{N} \alpha_n y_n \boldsymbol{x}_n$ in Lagrangian, we get the dual problem as (verify)

$$\max_{\boldsymbol{\alpha}\geq 0} \mathcal{L}_D(\boldsymbol{\alpha}) = \sum_{n=1}^{N} \alpha_n - \frac{1}{2} \sum_{m,n=1}^{N} \alpha_m \alpha_n y_m y_n (\boldsymbol{x}_m^T \boldsymbol{x}_n)$$

Solving Hard-Margin SVM

- Can write the objective more compactly in vector/matrix form as

$$\max_{\boldsymbol{\alpha} \geq 0} \quad \mathcal{L}_D(\boldsymbol{\alpha}) = \boldsymbol{\alpha}^\top \mathbf{1} - \frac{1}{2}\boldsymbol{\alpha}^\top \mathbf{G}\boldsymbol{\alpha}$$

  where $\mathbf{G}$ is an $N \times N$ matrix with $G_{mn} = y_m y_n \boldsymbol{x}_m^\top \boldsymbol{x}_n$, and $\mathbf{1}$ is a vector of 1s

- **Good news:** This is maximizing a concave function (or minimizing a convex function - verify that the Hessian is $\mathbf{G}$, which is p.s.d.). Note that our original SVM objective was also convex
- **Important:** Inputs $\boldsymbol{x}$'s only appear as inner products (helps to "kernelize"; more on this later)
- Can solve† the above objective function for $\boldsymbol{\alpha}$ using various methods, e.g.,
  - Treating the objective as a Quadratic Program (QP) and running some off-the-shelf QP solver such as `quadprog` (MATLAB), CVXOPT, CPLEX, etc.
  - Using (projected) gradient methods (projection needed because the α's are constrained). Gradient methods will usually be much faster than QP methods.
  - Using co-ordinate ascent methods (optimize for one α_n at a time); often very fast

† If interested in more details of the solver, see: "Support Vector Machine Solvers" by Bottou and Lin

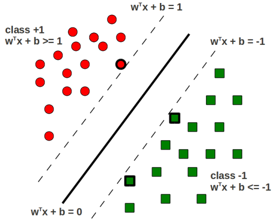
Hard-Margin SVM: The Solution

- Once we have the α_n's, $\boldsymbol{w}$ and b can be computed as:

$$\boldsymbol{w} = \sum_{n=1}^{N} \alpha_n y_n \boldsymbol{x}_n \quad \text{(we already saw this)}$$

$$b = -\frac{1}{2}\left(\min_{n:y_n=+1} \boldsymbol{w}^T\boldsymbol{x}_n + \max_{n:y_n=-1} \boldsymbol{w}^T\boldsymbol{x}_n\right) \quad \text{(exercise)}$$

- **A nice property:** Most α_n's in the solution will be zero (sparse solution)



- Reason: Karush-Kuhn-Tucker (KKT) conditions
- For the optimal α_n's

$$\alpha_n\{1 - y_n(\boldsymbol{w}^T\boldsymbol{x}_n + b)\} = 0$$

- α_n is non-zero only if $\boldsymbol{x}_n$ lies on one of the two margin boundaries, i.e., for which $y_n(\boldsymbol{w}^T\boldsymbol{x}_n + b) = 1$
- These examples are called support vectors
- Recall the support vectors "support" the margin boundaries

Solving Soft-Margin SVM

Solving Soft-Margin SVM

- Recall the soft-margin SVM optimization problem:

$$\min_{\boldsymbol{w},b,\boldsymbol{\xi}} \quad f(\boldsymbol{w},b,\boldsymbol{\xi}) = \frac{||\boldsymbol{w}||^2}{2} + C\sum_{n=1}^{N}\xi_n$$
$$\text{subject to} \quad 1 \leq y_n(\boldsymbol{w}^T\boldsymbol{x}_n + b) + \xi_n, \quad -\xi_n \leq 0 \qquad n = 1,\ldots,N$$

- Note: $\boldsymbol{\xi} = [\xi_1,\ldots,\xi_N]$ is the vector of slack variables
- Introduce Lagrange Multipliers α_n, β_n $(n = \{1,\ldots,N\})$, for constraints, and solve the Lagrangian:

$$\min_{\boldsymbol{w},b,\boldsymbol{\xi}} \max_{\boldsymbol{\alpha}\geq 0,\boldsymbol{\beta}\geq 0} \mathcal{L}(\boldsymbol{w},b,\xi,\alpha,\beta) = \frac{||\boldsymbol{w}||^2}{2} + C\sum_{n=1}^{N}\xi_n + \sum_{n=1}^{N}\alpha_n\{1 - y_n(\boldsymbol{w}^T\boldsymbol{x}_n + b) - \xi_n\} - \sum_{n=1}^{N}\beta_n\xi_n$$

- Note: The terms in red above were not present in the hard-margin SVM
- Two sets of dual variables $\boldsymbol{\alpha} = [\alpha_1,\ldots,\alpha_N]$ and $\boldsymbol{\beta} = [\beta_1,\ldots,\beta_N]$. We'll eliminate the primal variables $\boldsymbol{w}, b, \boldsymbol{\xi}$ to get dual problem containing the dual variables (just like in the hard margin case)

Solving Soft-Margin SVM

- The Lagrangian problem to solve

$$\min_{\boldsymbol{w},b,\boldsymbol{\xi}} \max_{\boldsymbol{\alpha}\geq 0,\boldsymbol{\beta}\geq 0} \mathcal{L}(\boldsymbol{w},b,\xi,\alpha,\beta) = \frac{\boldsymbol{w}^\top \boldsymbol{w}}{2} + +C\sum_{n=1}^{N}\xi_n + \sum_{n=1}^{N}\alpha_n\{1 - y_n(\boldsymbol{w}^T\boldsymbol{x}_n + b) - \xi_n\} - \sum_{n=1}^{N}\beta_n\xi_n$$

- Take (partial) derivatives of $\mathcal{L}$ w.r.t. $\boldsymbol{w}$, b, ξ_n and set them to zero

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{w}} = 0 \Rightarrow \boxed{\boldsymbol{w} = \sum_{n=1}^{N}\alpha_n y_n \boldsymbol{x}_n}, \quad \frac{\partial \mathcal{L}}{\partial b} = 0 \Rightarrow \sum_{n=1}^{N}\alpha_n y_n = 0, \quad \frac{\partial \mathcal{L}}{\partial \xi_n} = 0 \Rightarrow C - \alpha_n - \beta_n = 0$$

- Note: Solution of $\boldsymbol{w}$ again has the same form as in the hard-margin case (weighted sum of all inputs with α_n being the importance of input $\boldsymbol{x}_n$)
- Note: Using $C - \alpha_n - \beta_n = 0$ and $\beta_n \geq 0 \Rightarrow \alpha_n \leq C$ (recall that, for the hard-margin case, $\alpha \geq 0$)
- Substituting these in the Lagrangian $\mathcal{L}$ gives the Dual problem

$$\max_{\boldsymbol{\alpha}\leq C,\boldsymbol{\beta}\geq 0} \mathcal{L}_D(\boldsymbol{\alpha},\boldsymbol{\beta}) = \sum_{n=1}^{N}\alpha_n - \frac{1}{2}\sum_{m,n=1}^{N}\alpha_m\alpha_n y_m y_n(\boldsymbol{x}_m^T\boldsymbol{x}_n)$$

Solving Soft-Margin SVM

- Interestingly, the dual variables $\boldsymbol{\beta}$ don't appear in the objective!
- Just like the hard-margin case, we can write the dual more compactly as

$$\max_{\boldsymbol{\alpha} \leq C} \quad \mathcal{L}_D(\boldsymbol{\alpha}) = \boldsymbol{\alpha}^\top \mathbf{1} - \frac{1}{2}\boldsymbol{\alpha}^\top \mathbf{G}\boldsymbol{\alpha}$$

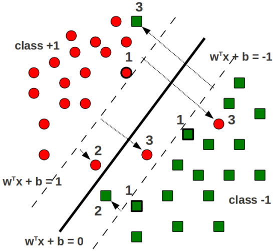
where $\mathbf{G}$ is an $N \times N$ matrix with $G_{mn} = y_m y_n \boldsymbol{x}_m^\top \boldsymbol{x}_n$, and $\mathbf{1}$ is a vector of 1s

- Like hard-margin case, solving the dual requires concave maximization (or convex minimization)
- Can be solved[†] the same way as hard-margin SVM (except that $\boldsymbol{\alpha} \leq C$)
  - Can solve for $\boldsymbol{\alpha}$ using QP solvers or (projected) gradient methods
- Given $\boldsymbol{\alpha}$, the solution for $\boldsymbol{w}, b$ has the same form as hard-margin case
- **Note:** $\boldsymbol{\alpha}$ is again sparse. Nonzero α_n's correspond to the support vectors

[†] If interested in more details of the solver, see: "Support Vector Machine Solvers" by Bottou and Lin

Support Vectors in Soft-Margin SVM

- The hard-margin SVM solution had only one type of support vectors
  - .. ones that lie on the margin boundaries $\boldsymbol{w}^T\boldsymbol{x} + b = -1$ and $\boldsymbol{w}^T\boldsymbol{x} + b = +1$
- The soft-margin SVM solution has three types of support vectors



1. Lying on the margin boundaries $\boldsymbol{w}^T\boldsymbol{x} + b = -1$ and $\boldsymbol{w}^T\boldsymbol{x} + b = +1$ ($\xi_n = 0$)
2. Lying within the margin region ($0 < \xi_n < 1$) but still on the correct side
3. Lying on the wrong side of the hyperplane ($\xi_n \geq 1$)

SVMs via Dual Formulation: Some Comments

- Recall the final dual objectives for hard-margin and soft-margin SVM

Hard-Margin SVM: $\max_{\boldsymbol{\alpha} \geq 0} \quad \mathcal{L}_D(\alpha) = \boldsymbol{\alpha}^\top \mathbf{1} - \frac{1}{2}\boldsymbol{\alpha}^\top \mathbf{G}\boldsymbol{\alpha}$

Soft-Margin SVM: $\max_{\boldsymbol{\alpha} \leq C} \quad \mathcal{L}_D(\alpha) = \boldsymbol{\alpha}^\top \mathbf{1} - \frac{1}{2}\boldsymbol{\alpha}^\top \mathbf{G}\boldsymbol{\alpha}$

- The dual formulation is nice due to two primary reasons:
  - Allows conveniently handling the margin based constraint (via Lagrangians)
  - Important: Allows learning nonlinear separators by replacing inner products (e.g., $G_{mn} = y_m y_n \boldsymbol{x}_m^\top \boldsymbol{x}_n$) by kernelized similarities (kernelized SVMs)
- However, the dual formulation can be expensive if N is large. Have to solve for N variables $\boldsymbol{\alpha} = [\alpha_1, \ldots, \alpha_N]$, and also need to store an $N \times N$ matrix $\mathbf{G}$
- A lot of work[†] on speeding up SVM in these settings (e.g., can use co-ord. descent for $\boldsymbol{\alpha}$)

[†] See: "Support Vector Machine Solvers" by Bottou and Lin

SVM: The Regularized Loss Function View

- Maximize the margin subject to constraints led to the soft-margin formulation of SVM

$$\arg\min_{\boldsymbol{w},b,\boldsymbol{\xi}} \frac{||\boldsymbol{w}||^2}{2} + C\sum_{n=1}^{N}\xi_n$$
$$\text{subject to} \quad y_n(\boldsymbol{w}^T\boldsymbol{x}_n + b) \geq 1 - \xi_n, \quad \xi_n \geq 0 \qquad n = 1,\ldots,N$$

- Note that the slack ξ_n is the same as $\max\{0, 1 - y_n(\boldsymbol{w}^\top\boldsymbol{x}_n + b)\}$, i.e., hinge loss for $(\boldsymbol{x}_n, y_n)$
- **Another View:** Thus the above is equivalent to minimizing the ℓ_2 regularized hinge loss

$$\mathcal{L}(\boldsymbol{w}, b) = \sum_{n=1}^{N}\max\{0, 1 - y_n(\boldsymbol{w}^\top\boldsymbol{x}_n + b)\} + \frac{\lambda}{2}\boldsymbol{w}^\top\boldsymbol{w}$$

- **Comparing the two:** Sum of slacks is like sum of hinge losses, C and λ play similar roles
- Can learn $(\boldsymbol{w}, b)$ directly by minimizing $\mathcal{L}(\boldsymbol{w}, b)$ using (stochastic)(sub)gradient descent
  - Hinge-loss version preferred for linear SVMs, or with other regularizers on $\boldsymbol{w}$ (e.g., ℓ_1)

Multiclass SVM

- Multiclass SVMs use K weight vectors $\mathbf{W} = [\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_K]$ (similar to softmax regression)

$$\hat{y}_* = \arg\max_k \boldsymbol{w}_k^\top \boldsymbol{x}_n \qquad \text{(predition rule)}$$

- Just like binary case, we can formulate a maximum-margin problem (without or with slacks)

$$\hat{\mathbf{W}} = \arg\min_{\mathbf{W}} \sum_{k=1}^{K} \frac{||\boldsymbol{w}_k||^2}{2} \qquad \text{s.t.} \quad \boldsymbol{w}_{y_n}^\top \boldsymbol{x}_n \geq \boldsymbol{w}_k^\top \boldsymbol{x}_n + 1 \quad \forall k \neq y_n$$

$$\hat{\mathbf{W}} = \arg\min_{\mathbf{W}} \sum_{k=1}^{K} \frac{||\boldsymbol{w}_k||^2}{2} + C \sum_{n=1}^{N} \xi_n \qquad \text{s.t.} \quad \boldsymbol{w}_{y_n}^\top \boldsymbol{x}_n \geq \boldsymbol{w}_k^\top \boldsymbol{x}_n + 1 - \xi_n \quad \forall k \neq y_n$$

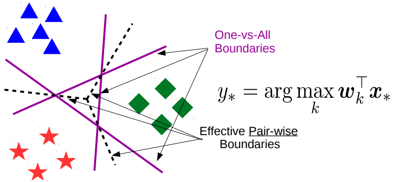
- Want score w.r.t. correct class to be at least 1 more than score w.r.t. all other classes
- The version with slack corresponds to minimizing a multi-class hinge loss

$$\boxed{\mathcal{L}(\mathbf{W}) = \max\{0, 1 + \max_{k \neq y_n} \boldsymbol{w}_k^\top \boldsymbol{x}_n - \boldsymbol{w}_{y_n}^\top \boldsymbol{x}_n\}} \qquad \text{(Crammer-Singer multiclass SVM)}$$

- Loss $= 0$ if score on correct class is at least 1 more than score on next best scoring class
- Can optimize these similar to how we did it for binary SVM

Multiclass SVM using Binary SVM?

- Can use binary classifiers to solve multiclass problems
- Note: These approaches can be used with other binary classifiers too (e.g., logistic regression)
- One-vs-All (also called One-vs-Rest): Construct K binary classification problems



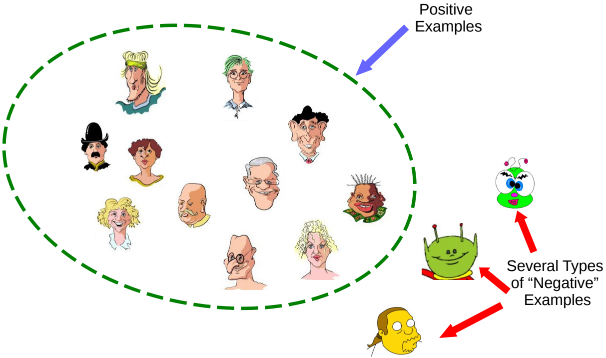
- All-Pairs: Learn K-choose-2 binary classifiers, one for each pair of classes (j, k)

$$y_* = \arg\max_k \sum_{j \neq k} \boldsymbol{w}_{j,k}^\top \boldsymbol{x}_* \qquad \text{(predict } k \text{ that wins over all others the most)}$$

- All-Pairs approach can be expensive at training and test time (but ways to speed up)

One-Class Classification

- Can we learn from examples of just one class, say positive examples?
- May be desirable if there are many types of negative examples

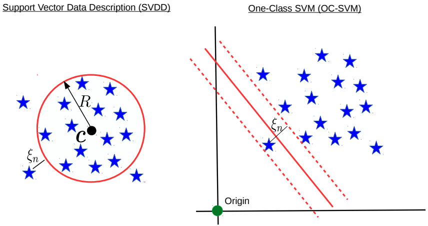


- "Outlier/Novelty Detection" problems can also be formulated like this

Figure credit: Refael Chickvashvili

One-Class Classification via SVM-like methods

- There are two popular SVM-type approaches to solve one-class problems



- Approach 1: Assume positives lie within a ball with smallest possible radius (and allow slacks)
  - Known as "Support Vector Data Description" (SVDD). Proposed by [Tax and Duin, 2004]
- Approach 2: Find a max-marg hyperplane separating positives from origin (representing negatives)
  - Known as "One-Class SVM" (OC-SVM). Proposed by [Schölkopf et al., 2001]
- Optimization problems for both cases can be solved similary as in binary SVM (e.g., via Lagrangian)

One-Class Classification via SVM-like methods

- There are two popular SVM-type approaches to solve one-class problems

Support Vector Data Description (SVDD)

$$\arg\min_{R,\boldsymbol{c},\xi} R^2 + \frac{1}{\nu N}\sum_{n=1}^{N}\xi_n$$
$$\text{s.t. } ||\boldsymbol{x}_n - \boldsymbol{c}||^2 \leq R^2 + \xi_n \quad \forall n$$
$$\xi_n \geq 0$$

Prediction Rule: $y_* = +1$ if $||\boldsymbol{x}_* - \boldsymbol{c}||^2 - R^2 < 0$

One-Class SVM (OC-SVM)

$$\arg\min_{\boldsymbol{w},\rho,\xi} ||\boldsymbol{w}||^2 + \frac{1}{\nu N}\sum_{n=1}^{N}\xi_n - \rho$$
$$\text{s.t. } \boldsymbol{w}^\top \boldsymbol{x}_n \geq \rho - \xi_n \quad \forall n$$
$$\xi_n \geq 0$$

Prediction Rule: $y_* = +1$ if $\boldsymbol{w}^\top \boldsymbol{x}_* > \rho$

- Approach 1: Assume positives lie within a ball with smallest possible radius (and allow slacks)
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- Approach 2: Find a max-marg hyperplane separating positives from origin (representing negatives)
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Nonlinear SVM ?

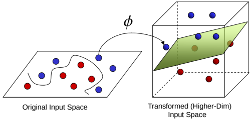
- A nice property of SVM (and many other models) is that inputs only appear as inner products
- For example, recall the dual problem for soft-margin SVM had the form

$$\arg\max_{\boldsymbol{\alpha} \leq C} \mathcal{L}_D(\boldsymbol{\alpha}) = \boldsymbol{\alpha}^\top \mathbf{1} - \frac{1}{2}\boldsymbol{\alpha}^\top \mathbf{G}\boldsymbol{\alpha}$$

  where $\mathbf{G}$ is an $N \times N$ matrix with $G_{mn} = y_m y_n \boldsymbol{x}_m^\top \boldsymbol{x}_n$, and $\mathbf{1}$ is a vector of 1s
- We can replace each inner-product by any general form of inner product, e.g.

$$k(\boldsymbol{x}_n, \boldsymbol{x}_m) = \phi(\boldsymbol{x}_n)^\top \phi(\boldsymbol{x}_m)$$

  .. where ϕ is some transformation (e.g., a higher-dimensional mapping) of the data



- Note: Often the mapping ϕ doesn't need to be explicitly computed ("kernel" magic - next class)!
- Can still learn a linear model in the new space but be nonlinear in the original space (wondeful!)

SVM: Some Notes

- A hugely (perhaps the most!) popular classification algorithm
- Reasonably mature, highly optimized SVM softwares freely available (perhaps the reason why it is more popular than various other competing algorithms)
  - Some popular ones: libSVM, LIBLINEAR, scikit-learn also provides SVM
- Lots of work on scaling up SVMs† (both large N and large D)
- Extensions beyond binary classification (e.g., multiclass, one-class, structured outputs)
- Can even be used for regression problems (Support Vector Regression)
  - The ϵ-insensitive loss for regression does precisely that!
- Nonlinear extensions possible via kernels (next class)

† See: "Support Vector Machine Solvers" by Bottou and Lin