A Geometric Approach to Graph Isomorphism

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Graph Isomorphism Problem (GIP)

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- Does there exist a bijection (permutation) $\sigma : V(G_1) \to V(G_2)$ s.t. $\{u, v\} \in E_1$ iff $\{\sigma(u), \sigma(v)\} \in E_2$?
- Not known to be either in P or NP-Complete.
- Best known algorithm runs in $2^{O(\sqrt{n \log n})}$ time.

An Integer Linear Program for GIP

IP-GI: Find a point $Y \in \{0,1\}^{n^2 \times n^2}$ subject to the following:

$$
\begin{aligned}
& Y_{ij,kl} - Y_{kl,ij} = 0 \ , \ \forall \ i,j,k,l && \text{(1a)} \\
& Y_{ij,il} = Y_{ji,li} = 0 \ , \ \forall \ i, \forall \ j \neq l && \text{(1b)} \\
& \sum_k Y_{ij,kl} = \sum_k Y_{ij,lk} = Y_{ij,ij} \ , \ \forall \ i,j,l && \text{(1c)} \\
& \sum_j Y_{ij,ij} = \sum_j Y_{ji,ji} = 1 \ , \ \forall \ i && \text{(1d)} \\
& Y_{ij,kl} = 0 \ , \ \{i,k\} \in E(G_1) \text{ and } \{j,l\} \notin E(G_2) \text{ or} \\
& \qquad \{i,k\} \notin E(G_1) \text{ and } \{j,l\} \in E(G_2) && \text{(1e)}
\end{aligned}
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$$\begin{aligned} Y_{ij,kl} = 0 \ , \ & \{i,k\} \in E(G_1) \text{ and } \{j,l\} \notin E(G_2) \text{ or} \\ & \{i,k\} \notin E(G_1) \text{ and } \{j,l\} \in E(G_2) \end{aligned} \tag{1e}$$

Theorem

IP-GI has a solution iff $G_1 \simeq G_2$.

Solution Points of IP-GI

Second-order Permutation Matrix, $P_\sigma^{[2]}$

- The $n^2 \times n^2$ symmetric matrix with $(P_\sigma^{[2]})_{ij,kl} = (P_\sigma)_{ij}(P_\sigma)_{kl}$.

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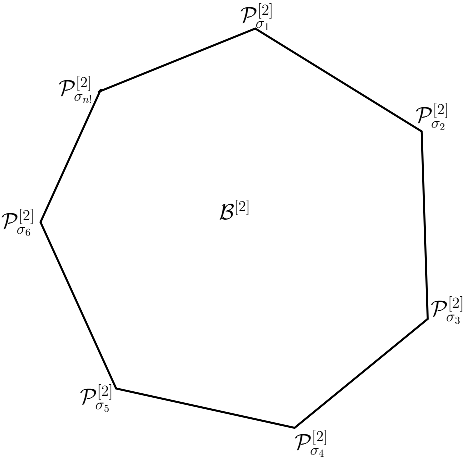
For example:

$$P_\sigma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, P_\sigma^{[2]} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Polytope $\mathcal{B}^{[2]}$

- **Convex hull of** $\mathcal{P}_{\sigma}^{[2]}\ \forall\ \sigma \in S_n$



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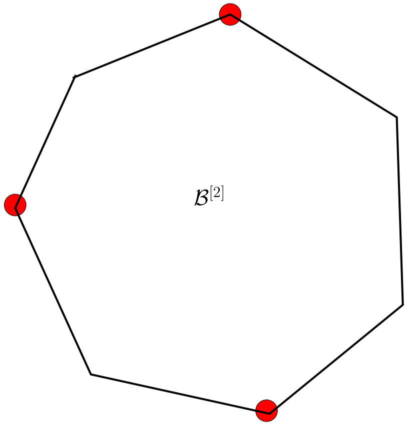

Figure: Red Vertices Correspond to the Isomorphisms Between G_1, G_2

Linear Programming Relaxation of IP-GI

$$\text{LP-GI:} \quad \text{Find a point} \quad Y$$
$$\text{subject to} \quad (1a)\text{-}(1e)$$
$$Y_{ij,kl} \geq 0 \ , \ \forall \ i, j, k, l \qquad (2a)$$

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Note

$Y_{ij,kl} \leq 1$ is implied.

Feasible Region of LP-GI

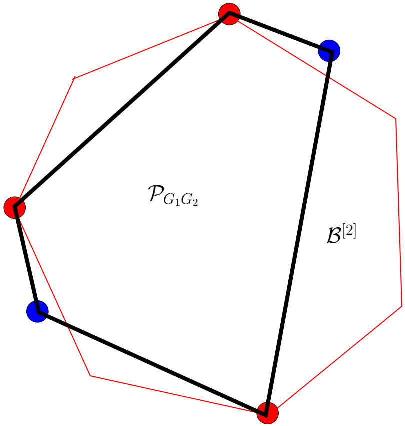


Figure: The Blue Vertices are the Non-Integral Vertices of Polytope $\mathcal{P}_{G_1 G_2}$

Polytope $\mathcal{P}$

- **Feasible region of LP-GI ($\mathcal{P}_{G_1G_2}$) when $G_1 = G_2 = (V, \emptyset)$ or $G_1 = G_2 = K_n$**

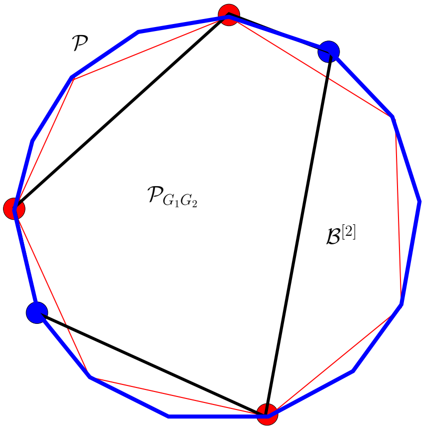


Figure: The Vertices in Blue are the Vertices of Polytope $\mathcal{P}$

Isomorphic Graphs

- **Graphs G_1, G_2 are isomorphic iff $\mathcal{P}_{G_1G_2} \cap \mathcal{B}^{[2]}$ is non-empty**

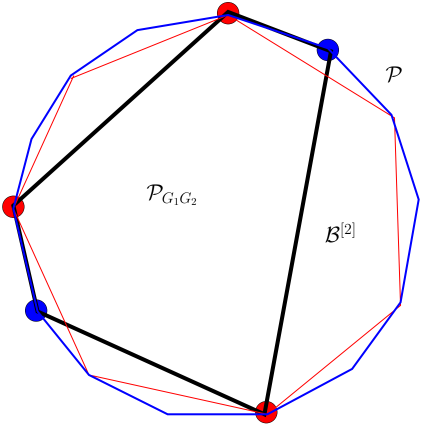


Figure: $\mathcal{P}_{G_1 G_2} \cap \mathcal{B}^{[2]}$ is Non-Empty

Non-Isomorphic Graphs

- $\mathcal{P}_{G_1G_2} \subseteq \mathcal{P} \setminus \mathcal{B}^{[2]}$ **when** $G_1 \not\cong G_2$

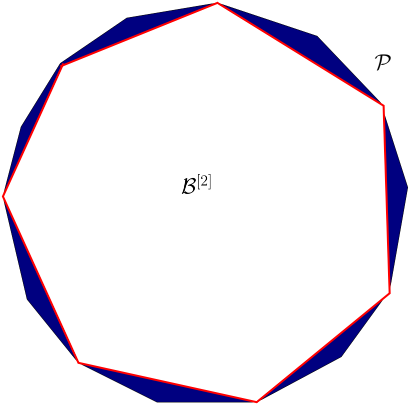
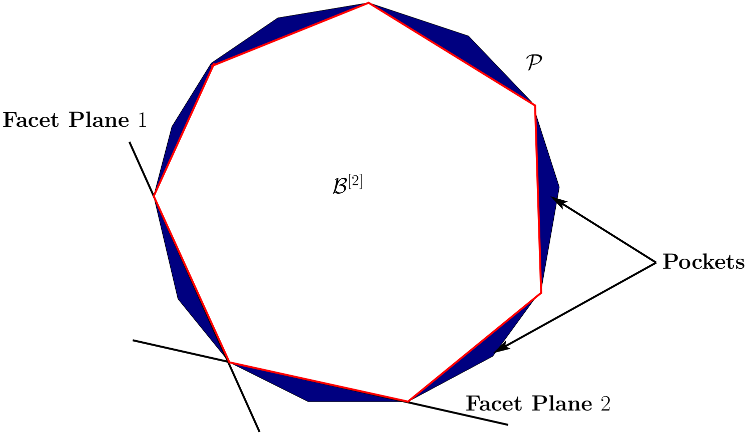


Figure: Region $\mathcal{P} \setminus \mathcal{B}^{[2]}$

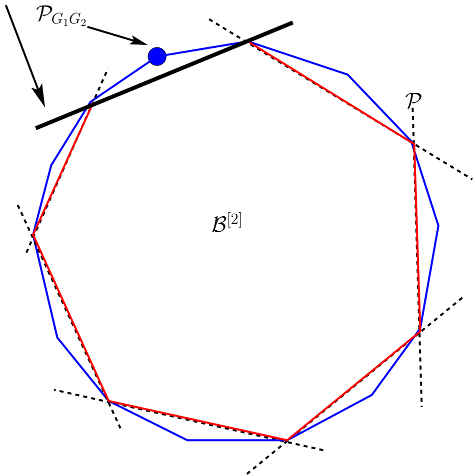
Pocket

- **Region of $\mathcal{P} \setminus \mathcal{B}^{[2]}$ on non-$\mathcal{B}^{[2]}$ side of a Facet plane of $\mathcal{B}^{[2]}$**
- **Pockets are disjoint**
- **$\mathcal{P}_{G_1G_2}$ must belong to a unique Pocket**

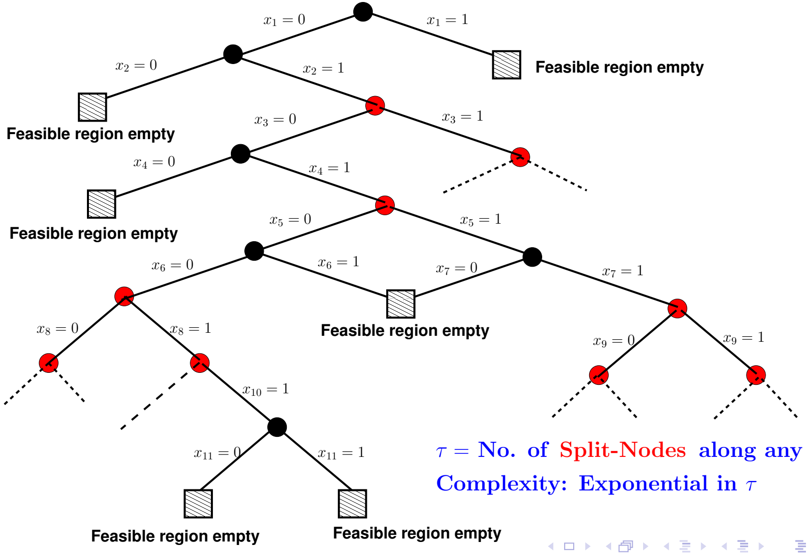


An Important Property for Non-Isomorphic Graphs

$\mathcal{P}_{G_1G_2}$ **violates a unique Facet Plane of** $\mathcal{B}^{[2]}$



The Algorithm



τ = No. of Split-Nodes along any path

Complexity: Exponential in τ

First Family of Facets

Theorem

$Y_{i_1j_1,kl} + Y_{i_2j_2,kl} + \ldots + Y_{i_mj_m,kl} \leq Y_{kl,kl} + \sum_{r \neq s} Y_{i_rj_r,i_sj_s}$, where $i_1, \ldots, i_m, k$ are all distinct and $j_1, \ldots, j_m, l$ are also distinct. In addition, $n \geq 6, m \geq 3$.

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Lemma

$\tau = 0$ when the feasible region lies in a pocket defined by any facet above with $m > 3$.

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Lemma

$\tau = 0$ when the feasible region lies in a pocket defined by any facet above with $m > 3$.

Note

The case of $m = 3$ is handled by explicitly adding the corresponding *polynomially many* inequalities to LP-GI.

Second Family of Facets

Theorem (Jünger-Kaibel)

$$-(\beta-1)\sum_{(ij)\in P\times Q} Y_{ij,ij} + \sum_{(ij)\neq(kl)\in P\times Q, i<k} Y_{ij,kl} + \frac{\beta^2-\beta}{2} \geq 0, \text{ where}$$

$P, Q \subseteq [n]$; $\beta+1 \leq |P|, |Q| \leq n-3$; $|P|+|Q| \leq n-3+\beta$; $\beta \geq 2$.

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$\tau = 0$ when the feasible region lies in a pocket defined by any facet above with $|P| > \beta+1$ or $|Q| > \beta+1$ or $|P| = |Q| = \beta+1, \beta > 2$.

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Lemma

$\tau = 0$ when the feasible region lies in a pocket defined by any facet above with $|P| > \beta + 1$ or $|Q| > \beta + 1$ or $|P| = |Q| = \beta + 1, \beta > 2$.

Note

The case of $\beta = 2$ and $|P| = |Q| = 3$ is handled by explicitly adding the corresponding *polynomially many* inequalities to LP-GI.

Third Family of Facets

Theorem (Jünger-Kaibel)

In the following $P_1, P_2, Q \subseteq [n]$, $P_1 \cap P_2 = \emptyset$:

$$-(\beta - 1) \sum_{(ij)\in P_1\times Q} Y_{ij,ij} + \beta \sum_{(ij)\in P_2\times Q} Y_{ij,ij} + \sum_{(ij)\neq(kl)\in P_1\times Q, i<k} Y_{ij,kl}$$

$$+ \sum_{(ij)\neq(kl)\in P_2\times Q, i<k} Y_{ij,kl} - \sum_{(ij)\in P_1\times Q,(kl)\in P_2\times Q} Y_{ij,kl} + \frac{\beta^2 - \beta}{2} \geq 0.$$

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$$+\sum_{(ij)\neq(kl)\in P_2\times Q, i<k} Y_{ij,kl} - \sum_{(ij)\in P_1\times Q,(kl)\in P_2\times Q} Y_{ij,kl} + \frac{\beta^2-\beta}{2} \geq 0.$$

Lemma

$\tau = 0$ *when the feasible region lies in a pocket defined by any facet above subject to: (i)* $|P_1|, |P_2| \geq 3$, *(ii) if* $\beta \geq 0$ *and* $\min\{|Q|, |P_1|\} \geq \beta+1$ *then* $|Q|+|P_1|+3 \leq n+\beta$, *(iii) if* $\beta < 0$ *and* $\min\{|Q|, |P_2|\} \geq 2-\beta$ *then* $|Q|+|P_2|+3 \leq n+1-\beta$.

Fourth Family of Facets

Theorem

$Y_{p_1q_1,kl} + Y_{p_2q_2,kl} + Y_{p_1q_2,kl} \leq Y_{kl,kl} + Y_{p_1q_1,p_2q_2}$, where p_1, p_2, k are distinct and q_1, q_2, l are also distinct and $n \geq 6$.

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Note

The above are polynomially many and can be included in LP-GI.

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Note

The above are polynomially many and can be included in LP-GI.

Modified Linear Program

LP-GI with the base case facets and the above family of facets included is referred to as *modified* LP-GI.

Trivial Facets

Theorem (Jünger-Kaibel)

$Y_{ij,kl} \geq 0$ for every i, j, k, l such that $i \neq k$ and $j \neq l$.

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Theorem (Jünger-Kaibel)

$Y_{ij,kl} \geq 0$ *for every* i, j, k, l *such that* $i \neq k$ *and* $j \neq l$.

Note

The above are already part of LP-GI.

The Main Theorem

Theorem

The Algorithm, using modified LP-GI, detects non-isomorphic graph pairs in polynomial time if the solution is confined to a pocket due to any of the known facets.

$GI \in P$?

Yes, if the known facets are all the facets of $\mathcal{B}^{[2]}$.

A General Inequality

All the known facets of $\mathcal{B}^{[2]}$ are special instances of a general inequality

$$\sum_{ijkl} n_{ij} n_{kl} Y_{ij,kl} + (\beta - 1/2)^2 \geq (2\beta - 1) \sum_{ij} n_{ij} Y_{ij,ij} + 1/4$$

where $\beta \in \mathbb{Z}$ and $n_{ij} \in \mathbb{Z}$ for all (ij).

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where $\beta \in \mathbb{Z}$ and $n_{ij} \in \mathbb{Z}$ for all (ij).

There are more Facets

Theorem

There exists at least one facet of $\mathcal{B}^{[2]}$ which is not an instance of the above inequality.

Future Direction

- Find the remaining facets of $\mathcal{B}^{[2]}$ and show that $\tau = O(\log n)$ for them.

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- Find the remaining facets of $\mathcal{B}^{[2]}$ and show that $\tau = O(\log n)$ for them.
- Give a general inequality that includes all the facets.

Thank you!
Questions?