Partial Degree Bounded Edge Packing Problem with Arbitrary Bounds

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- Zhang showed PDBEP is NP-Hard even for uniform $c_v = 1$.

Our Contribution

Zhang, 2012

For uniform $c_v = k$:

- 2-approx for $k = 1$.

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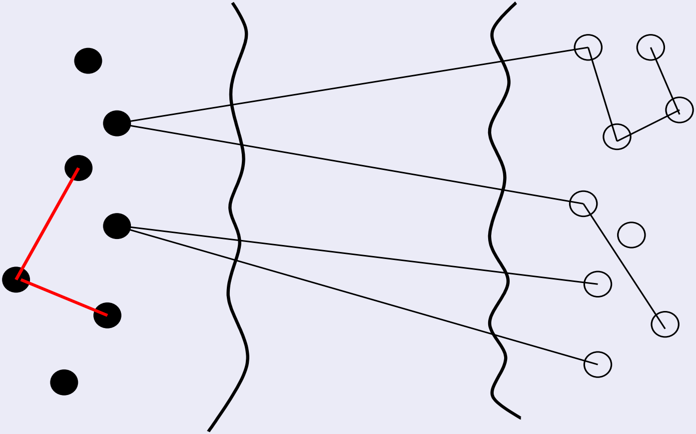
- $O(n \log n)$-time exact algorithm on trees.
- $2 + \log_2 n$-approx with weights on edges.

Upper Bound

Upper Bound on OPT



$OPT \leq \sum_{w \in W} c_w \leq \sum_{v \in V} c_v$, where $W = \{v \in V | d'_v \leq c_v\}$.

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Edge Deletion based Algorithm

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- Hence $|Y| \geq \sum_v c_v/2 \geq OPT/2$.

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$2 + \log_2 n$-Approximation

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- Then the heaviest solution gives a $2 + \log_2 |V|$ approximation of the problem.

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Exact Algorithm

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- Thus at each vertex we incur $O(|Ch| \log |Ch|)$ cost.
- That gives an overall time complexity of $O(n \log n)$.

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- However, $OPT \leq n$.
- $\implies$ integrality gap $= \Omega(n)$.

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Lagrangian Like Relaxation

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Lemma

Every maximal solution of IP2 is also a feasible solution of PDBEP.

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If $c_v \geq (1-\beta)d_v \, \forall v$, then any α approximate solution of IP2, which is also maximal, is a $2\alpha/(2-(1+\epsilon)\beta)$ approximation of PDBEP.

LP Relaxation

LP Relaxation of IP2

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Lemma

In a corner solution of LP2 on a non-empty graph there is at least one edge e with $y_e = 0$ *or* $y_e \geq 1/2$.

Algorithm using LP2

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The Iterative Rounding Algorithm returns a feasible solution of PDBEP.

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- Generalization to hypergraphs.

Thanks!