Diplomarbeit

Sylvester-Gallai-Konfigurationen und Verzweigte Überlagerungen

(From Sylvester-Gallai Configurations to Branched Coverings)

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Aus Siegen

Niemals aufgeben, niemals kapitulieren.

Deutsche Einleitung

Eine endliche Punktmenge X im projektiven Raum über einem Körper $\Bbbk$ heißt Sylvester-Gallai-k-Konfiguration, wenn keine lineare Untervarietät der Dimension $k-1$ genau k Schnittpunkte mit ihr hat. Wir schreiben auch kurz SG$_k$C. Das Resultat [Han] von Sten Hansen aus dem Jahr 1965 besagt, dass jede SG$_k$C über $\Bbbk = \mathbb{R}$ einen linearen Raum aufspannt, dessen Dimension durch $2k-3$ beschränkt ist.

Für $\Bbbk = \mathbb{C}$ wurde von Leroy Kelly in seiner Arbteit [Kel] gezeigt, dass eine SG$_2$C höchstens eine komplexe Ebene aufspannen kann. Es bleibt eine offene Frage, ob ein Resultat für $k > 2$ im Sinne Hansens auch über $\mathbb{C}$ formuliert werden kann. Auch über Körpern endlicher Charakteristik gibt es bisher nur wenig zufriedenstellende Dimensionsschranken.

Abgesehen von geometrischer Neugier wären solche Schranken auch von großem Interesse für die Komplexitätstheorie, da sie neue Ergebnisse im Bereich des Polynomial Identity Testing liefern würden.

Obgleich es für die Aussage im Falle $k = 2$ inzwischen kürzere und elementarere Beweise gibt, etwa [EPS], interessieren wir uns für die Methoden von Kelly's Beweis: Mittels geometrischer Dualität konnte er sich das Ergebnis [Hir, Theorem 3.1] von Hirzebruch über Geradenkonfigurationen in der komplex-projektiven Ebene zu Nutze machen. Dieses Ergebnis entstand als Nebenprodukt des Studiums komplexer Flächen. In [Hir] konstruierte Hirzebruch konstant verzweigte Überlagerungen der komplex-projektiven Ebene, um Flächen von allgemeinem Typ mit speziellen Invarianten zu konstruieren. In dem Buch [BHH] wird diese Konstruktion im Detail erläutert.

In dieser Arbeit verallgemeinern wir Hirzebruchs Methoden signifikant. Wir studieren "konstant verzweigte" Überlagerungen $Y \to X$ zwischen Varietäten beliebiger Dimension über einem algebraisch abgeschlossenen Grundkörper $\Bbbk$. Wir zeigen, dass die Singularitäten der Überlagerungsvarietät stets durch eine einfach zu charakterisierende Sequenz von Aufblasungen aufgelöst werden können. Es werden Formeln hergeleitet, um Selbstschnittzahl eines kanonischen Divisors und Euler-Charakteristik von Y und X miteinander in Verbindung zu bringen.

Zu jeder "strikten" Konfiguration von Hyperflächen konstruieren wir eine assoziierte Überlagerung von nicht-singulären Varietäten mit frei wählbarem Verzweigungsindex, deren Eigenschaften stark mit den kombinatorischen Daten der Konfiguration zusammen hängen. Jede Konfiguration von Hyperebenen im projektiven Raum wird strikt in diesem Sinne sein, und mit Hilfe geometrischer Dualität erhalten wir daher eine Methode, um jeder endlichen Punktmenge in $\mathbb{P}^s_{\mathbb{k}}$ eine verzweigte Überlagerung zuzuordnen.

Wir zeigen als Anwendungsbeispiel, wie Hirzebruchs Ergebnis als Spezialfall dieser Methoden entsteht. Wir erinnern daran, dass die Euler Charakteristik und der Kanonische Divisor einer Varietät der obersten und untersten Chern Klasse des Tangentialbündels entsprechen. Die Miyaoka-Yau Ungleichung stellt eine Beziehung zwischen diesen Größen her, aus der wir das Schlüsselargument für Kelly's Beweis ableiten.

Abschließend zitieren wir verwandte Ungleichungen in höheren Dimensionen und für den Fall positiver Charakteristik. Dies eröffnet Perspektiven für das Studium von Sylvester-Gallai Schranken anhand der zugehörigen, konstant verzweigten Überlagerungen.

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Introduction

History of Sylvester-Gallai Configurations

A complex cubic curve has nine points of inflection which form a rather curious configuration: The line defined by any two of them will intersect the curve in a third inflection point. Arguably, this observation motivated James Sylvester to ponder on similar configurations in real space. In 1893, he published the challenge [Syl], conjecturing that any finite set of points with *real* coordinates and the above property had to be colinear.

The first documented proof of this conjecture was given 1933 by Tibor Gallai. A configuration of finitely many points in projective space, such that there exists no line that passes through exactly two of them, is nowadays called a *Sylvester-Gallai Configuration*, or SGC for short.

In 1966, Jean-Pierre Serre conjectured that a complex SGC had to be coplanar, i.e. confined to a complex plane. A surprising proof was given in 1986 by Leroy Kelly in his paper [Kel]. Via geometric duality, he leveraged the seemingly unrelated result [Hir, Theorem 3.1] by Hirzebruch about line arrangements in the complex projective plane. The latter arose naturally from the study of minimal surfaces of general type. Since then, more elementary proofs have been devised and even generalized to quaternions, see [EPS].

A whole new and different generalization was introduced by Sten Hansen in 1965. He studied the dimension of the linear space spanned by a finite set $X \subseteq \mathbb{P}^s_{\mathbb{R}}$ under the condition that no linear subvariety of dimension $k-1$ intersects X in k points. For $k = 2$, it is the original Sylvester-Gallai Theorem that limits this dimension to 1. Motzkin had already established in his 1951's paper [Mot] that for $k = 3$, the dimension is bounded by 3. In [Han], Hansen proves that for general k, the bound on the dimension is $2k - 3$.

It is natural to ask whether the complex Sylvester-Gallai theorem can be generalized in a similar way. Furthermore, there are quite modern applications that raise the same question. We elaborate on some of them now.

SGCs and Polynomial Identity Testing

It is probably one of *the* most important problems on the verge of algebra and computer science to check for equality of two polynomials, given by either a black-box interface or arithmetic circuits. Since subtraction is usually an easy operation, it is equivalent to ask whether a given polynomial is the zero polynomial. Consequently, if an arithmetic circuit represents the zero polynomial, it is called an *identity*.

Depth-3 Polynomial Identity Testing

Instance: Natural numbers $k, d, n \in \mathbb{N}$. For $1 \leq i \leq k$ and $1 \leq j \leq d$, homogeneous linear polynomials $\ell_{ij} \in \Bbbk[x_1, \ldots, x_n]_1$.

Task: Decide whether $\sum_{i=1}^{k} \prod_{j=1}^{d} \ell_{ij}$ is the zero polynomial.

Remark : We need one layer of addition gates with fan-in n to construct the ℓ_{ij}, then a second layer of multiplication gates with maximal fan-in d and in the third layer, a single addition gate with fan-in k. We refer to this as a $\Sigma\Pi\Sigma(k, d, n)$-circuit.

Even in the above case of depth-3 circuits, progress has been stale. Just recently in 2009, the influential paper [KS2] gave a solution for $\Bbbk = \mathbb{Q}$. But let us start a little earlier in 2006, when a new numerical quantity began to play a role in the study of depth-3 circuits.

The *rank* of a circuit roughly measures the number of free variables: If a $\Sigma\Pi\Sigma(k, d, n)$-circuit has rank r, then there exists a linear transformation converting it into a $\Sigma\Pi\Sigma(k, d, r)$-circuit which is quite easy to determine. In [DS], Dvir and Shpilka observed that the rank of an identity is always very small and conjectured that it is polynomial in k.

It was Karnin and Shpilka in 2008 [KS1], who showed how small rank bounds for identities imply efficient black-box PIT algorithms. This fundamental result steeled the resolve to investigate rank bounds. In 2009, Kayal and Saraf [KS2] made a significant leap forward by proving a rank bound that was independent of d. What they had found and tapped into was the fact that Sylvester-Gallai Configurations are confined to low dimensions. The conjecture of [DS] was finally proven correct by Saxena and Seshadhri in 2010. A rank bound of $O(k^2)$ is given in their paper [SS].

Since all of this late progress is based on Hansen's result for real SGC's, it was repeatedly conjectured, by [KS2] and [SS], that it should be possible to obtain a similar result over $\mathbb{C}$. It is the goal of this thesis to give perspectives on how to tackle the problem. We are going to revisit Kelly's proof and generalize

Hirzebruch's algebraic geometry constructions to higher dimensions.

Overview and Thesis Outline

In Chapter 1, we will recall several well-known geometric preliminaries and motivate the later chapters. We introduce the concept of geometric duality in projective space. Anticipating the results of Chapter 3, we deduce Kelly's proof of the complex Sylvester-Gallai Theorem. As preparation for the chapters to come, we give an in-depth treatment of the blowup construction for algebraic varieties. Furthermore, we give a brief summary of intersection theory, closing with the definition of Chern classes and their relevant properties.

The heart of the thesis lies in Chapter 2. In 1983's paper [Hir], Hirzebruch constructed branched coverings of the complex projective plane that were associated to line arrangements. In 1987, the book [BHH] elaborated on the construction in greater detail, but remained limited to coverings of the complex plane. The construction we give in Section 2.6 constitutes a significant generalization of these ideas. We study the class of "constantly" branched coverings $Y \to X$ between varieties of any dimension and calculate formulas to relate the Euler characteristic and canonical divisors of X and Y. We prove a special desingularization result in Section 2.5. Using it, we are able to associate a covering of nonsingular varieties to any suitable arrangement of (sub)varieties which constantly branches to a degree of our choice.

This is done in the language of modern algebraic geometry and over an arbitrary, algebraically closed field. In particular, the construction works in positive characteristic if we add the mild assumption of tame ramification.

In particular, the theory developed in Chapter 2 provides a framework to construct nonsingular coverings that branch along hyperplane arrangements in projective space, whose properties reflect the combinatorial properties of the arrangement. By geometric duality, studying hyperplane arrangements is equivalent to studying finite sets of points – in our case, SGCs.

In Chapter 3, we show how Hirzebruch's result about line arrangements arises as a special case from our construction. We show that the Euler characteristic and canonical divisor of a variety correspond to the top and bottom Chern class, respectively. We use a famous inequality of Chern classes and our formulas from Chapter 2 to deduce relations between the combinatorial data of the arrangement. This yields the key argument that is required for Kelly's proof of the complex Sylvester-Gallai Theorem. We also obtain several intermediate results about constantly branched coverings between surfaces.

Finally, we outline possible further steps in Chapter 4. More precisely, we cite inequalities involving Chern classes in higher dimension and positive characteristic that appear promising for advancing Sylvester-Gallai bounds by means of the techniques from Chapter 2.

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Chapter 1

Preliminaries

In Section 1.1, we recall some terminology from graduate courses in Algebraic Geometry and set up notation. Then, after explaining the concept of geometric duality, we will give Kelly's proof of the Sylvester-Gallai Theorem right away in Section 1.2, anticipating the results of Chapter 3. This should serve as motivation, presenting an application for the ensuing theory. Section 1.3 contains a thorough introduction to the construction of blowing up, since a firm grasp on it is required to perform the desingularization in Section 2.5. Finally, we give a brief summary of intersection theory in Section 1.4 in order to introduce Chern classes and their basic properties.

1.1 Notions of Algebraic Geometry

We recall some notions of algebraic geometry. Our main references are [Har] and [Liu]. If X is a ringed space, we denote by $\mathcal{O}_X$ the associated sheaf of rings and by $\mathrm{sp}(X)$ the underlying topological space. For a point $P \in X$, we denote by $\mathcal{O}_{X,P}$ the **stalk at** P. For an open subset $U \subseteq X$, we write $\mathcal{O}_X(U)$ for the ring on U. If $\phi : X \to Y$ is a morphism of ringed spaces, $\mathscr{E}$ a sheaf of $\mathcal{O}_X$-modules and $\mathscr{F}$ a sheaf of $\mathcal{O}_Y$-modules, we define the **push-forward** $\phi_*(\mathscr{E})$ to be the sheaf on Y which satisfies $\phi_*(\mathscr{E})(V) = \mathscr{E}(\phi^{-1}(V))$ for all open $V \subseteq Y$. On the other hand, we also define a sheaf $\phi^{-1}(\mathscr{F})$ on X as the one associated to the presheaf

$$U \longmapsto \varinjlim_{\phi(U) \subseteq V} \mathscr{F}(V).$$

Then, the **pull-back** of $\mathscr{F}$ via ϕ is defined as $\phi^*(\mathscr{F}) := \phi^{-1}(\mathscr{F}) \otimes_{\phi^{-1}(\mathcal{O}_Y)} \mathcal{O}_X$.

An **affine scheme** is a locally ringed space X whose underlying topological space $X = \mathrm{Spec}(A)$ is the set of prime ideals of a commutative ring A, endowed with the Zariski topology. In other words, a subset of X is closed if and only if it is the **vanishing set**

$$Z(I) := \{ P \in \mathrm{Spec}(A) \mid P \supseteq I \}$$

of an ideal $I \subseteq A$. For $f \in A$, we let $A_f = A[f^{-1}]$ denote the localization of A by f. Then, $D(f) := \mathrm{Spec}(A) \setminus Z(f)$ is the open set where f does not vanish and we additionally require that $\mathcal{O}_X(D(f)) = A_f$. We denote by A_P the localization of A by the multiplicatively closed set $A \setminus P$ for any prime ideal $P \subset A$. It then follows that $\mathcal{O}_{X,P} \cong A_P$. If $Z \subseteq X = \mathrm{Spec}(A)$ is a subset, we denote by $I(Z) := \bigcap_{P \in Z} P$ the associated ideal. Hence,

$$I(Z(I)) = \sqrt{I} := \{ f \in A \mid \exists n \in \mathbb{N} : f^n \in I \}$$

is the **radical of** I. If M is an A-module, we denote by $M^{\sim}$ the (quasi-coherent) sheaf of $\mathcal{O}_X$-modules associated to M.

A **scheme** is a locally ringed space which can be covered by affine schemes. For a point $P \in X$, we denote by $\mathfrak{m}_P \subset \mathcal{O}_{X,P}$ the unique maximal ideal of the local ring $\mathcal{O}_{X,P}$. The field

$$\Bbbk(P) := \mathcal{O}_{X,P}/\mathfrak{m}_P$$

is called the **function field**[1] **of** P. If $Z := \overline{P}$ is the closure of P, we also write $\Bbbk(Z)$ instead of $\Bbbk(P)$. For a scheme X, we denote by X_{red} the associated reduced scheme. A **closed point** $P \in X$ is a point such that its closure contains only P itself, i.e. $\overline{P} = \{ P \}$.

A morphism of schemes $\varphi = (\varphi, \varphi^\sharp) : Y \to X$ consists of a topological component $\varphi : \mathrm{sp}(Y) \to \mathrm{sp}(X)$ and a morphism of sheaves $\varphi^\sharp : \mathcal{O}_X \to \varphi_* \mathcal{O}_Y$. The morphism is **finite** if for every affine open subset U of X, the subset $V := \varphi^{-1}(U)$ is affine and $\varphi^\sharp_U$ turns $\mathcal{O}_Y(V)$ into a finitely generated $\mathcal{O}_X(U)$-module. A **closed** (resp. **open**) **immersion** is a morphism $\iota : Z \to X$ whose topological component is an injective map onto a closed (resp. open) subset of X and $\iota^\sharp_P : \mathcal{O}_{X,\iota(P)} \to \mathcal{O}_{Z,P}$ is surjective (resp. an isomorphism) at every point $P \in Z$. For intuition in the affine case $X = \mathrm{Spec}(A)$, one should think of $\iota^\sharp$ as the canonical surjection from A to the coordinate ring A/I of the closed subvariety $Z(I)$.

If $\mathcal{I}$ is a quasi-coherent sheaf of ideals on X, we denote by $\mathcal{Z}(\mathcal{I})$ the closed subscheme associated to it. Correspondingly, if $Z \subseteq X$ is a closed subscheme,

[1]Note that this is usually called a *residue field*. However, it coincides with the function field of the induced reduced scheme on the closure of that point, so we don't make that distinction.

we denote by $\mathcal{I}(Z)$ the associated sheaf of ideals. If $Y \subseteq X$ is another closed subscheme, then

$$Z \sqcap Y := \mathcal{Z}(\mathcal{I}(Z) + \mathcal{I}(Y))$$

is called the **scheme-theoretic intersection** of Z and Y. We write

$$Z \cap Y := (Z \sqcap Y)_{\mathrm{red}}.$$

Quite generally, closed subsets $Z \subseteq X$ of a scheme X, when interpreted as a closed subscheme, are *usually* endowed with the induced reduced scheme structure – unless otherwise stated, as in the above cases.

If $S = \bigoplus_{d \geq 0} S_d$ is a graded ring, we denote by $X = \mathrm{Proj}(S)$ the set of its homogeneous prime ideals and endow it with the Zariski topology similar to the affine case, i.e. the closed subsets are of the form

$$Z_*(I) := \{\, P \in \mathrm{Proj}(S) \mid P \supseteq I \,\}$$

for a homogeneous ideal $I \subseteq S$. For $Z \subseteq X$, we denote by $I_*(Z) := \bigcap_{P \in Z} P$ the associated homogeneous ideal. If $f \in S$ is a homogeneous element, the open set where f does not vanish is $D_*(f) := \mathrm{Proj}(S) \setminus Z_*(f)$ and we require that $\mathcal{O}_X(D(f)) = (S_f)_0$. This turns $X = \mathrm{Proj}(S)$ into a scheme with local rings $\mathcal{O}_{X,P} = (S_P)_0$.

A **variety** is an integral, separated scheme of finite type over some algebraically closed field $\mathbb{k}$. Prominent examples are $\mathrm{Spec}(A)$ and $\mathrm{Proj}(S)$ for finitely generated, integral $\mathbb{k}$-algebras A and S. For two schemes X and Y over a common base scheme S, we denote by $X \times_S Y$ their **fiber product**. If X and Y are varieties over $\mathbb{k}$, we write $X \times Y$ instead of $X \times_{\mathrm{Spec}(\mathbb{k})} Y$. A **rational map** $\varphi : X \dashrightarrow Y$ between varieties is an equivalence class of morphisms $\varphi_U : U \to Y$ defined on nonempty open subsets $U \subseteq X$ such that

$$\varphi_U|_{U \cap V} = \varphi_V|_{U \cap V}.$$

If $R = A[x_0, \ldots, x_s]$ is the polynomial ring in $s+1$ variables over a commutative ring A, we denote by $\mathbb{P}^s_A := \mathrm{Proj}(R)$ the **projective s-space** over A. In particular, if $A = \mathbb{k}$ is a fixed base field, we usually write $\mathbb{P}^s$ instead of $\mathbb{P}^s_{\mathbb{k}}$. Any closed subvariety $Z \subseteq \mathbb{P}^s$ has a well-defined degree $\deg(Z)$, see [Har, I.7.6]. The closed points of $\mathbb{P}^s$ can be written in **projective coordinates** as

$$[a_0 : \ldots : a_s] := \left(x_i a_j - x_j a_i \;\middle|\; 0 \leq i, j \leq s \right) \in \mathrm{Proj}(R).$$

This identifies the closed points of $\mathbb{P}^s$ with $\mathbb{P}(\mathbb{k}^{s+1})$. Here, $\mathbb{P}(V)$ denotes the projectivization of any $\mathbb{k}$-vector space V, i.e.

$$\mathbb{P}(V) := (V \setminus \{\, 0 \,\})/\mathbb{k}^\times$$

where $\Bbbk^\times$ acts on $V \setminus \{ 0 \}$ by scalar multiplication.

A **linear subvariety** $L \subseteq \mathbb{P}^s$ is a subvariety with $\deg(L) = 1$. It is also called a d-**flat**, where $d = \dim(L)$. In particular, $I_*(L)$ is generated in degree one and there exists a unique subspace $W \subseteq \Bbbk^{s+1}$ such that the closed points of L correspond to $\mathbb{P}(W)$. We write $L = \mathbb{P}(W)$ by abuse of notation. If $L' = \mathbb{P}(W')$ is another linear subvariety, we define their **linear span** to be

$$L + L' := \mathbb{P}(W + W').$$

Finally, we call $\mathbb{A}^s := \mathbb{A}^s_\Bbbk := \operatorname{Spec}(\Bbbk[x_1, \ldots, x_s])$ the **affine** s**-space** over $\Bbbk$.

1.2 Sylvester-Gallai Configurations

We first give a brief introduction to the concept of geometric duality in the projective space $\mathbb{P}^s = \mathbb{P}^s_\Bbbk$ over some field $\Bbbk$. For the special case $s = 2$, it yields an incidence-preserving one-to-one correspondence between lines and points. Generalizing to arbitrary s, it is an inclusion-reversing (and hence, incidence-preserving) one-to-one correspondence between the linear subvarieties of codimension d and those of dimension $d - 1$.

We begin by recalling a well-known fact of linear algebra from the theory of bilinear forms:

Fact/Definition 1.1 (Geometric Dual). *Let U be a $\Bbbk$-vector space of finite dimension with a nondegenerate, symmetric, bilinear form*

$$\langle -, - \rangle : U \times U \longrightarrow \Bbbk.$$

If $V \subset U$ is a subspace, its ***geometric dual*** *is*

$$V^\perp := \{ u \in U \mid \forall v \in V : \langle u, v \rangle = 0 \}.$$

Then, if W is another subspace of U,

(a). $\dim(W^\perp) = \dim(U) - \dim(W)$.

(b). If $W \subseteq V$, then $W^\perp \supseteq V^\perp$.

(c). $W^{\perp\perp} = W$.

(d). $(W + V)^\perp = W^\perp \cap V^\perp$.

Proof. Part (a) is well-known linear algebra, see [MH, 3.1]. Since $W \subseteq W^{\perp\perp}$, part (c) follows because

$$\dim(W^{\perp\perp}) = \dim(U) - \dim(W^\perp) = \dim(W).$$

For part (b), assume that $W \subseteq V$ and $u \in V^{\perp}$. In other words, $\langle u, v \rangle = 0$ for all $v \in V$. In particular, $\langle u, w \rangle = 0$ for all $w \in W \subseteq V$, so $u \in W^{\perp}$. Part (d) can also be verified by elementary means:

$$\begin{aligned}(V+W)^{\perp} &= \{ u \in U \mid \forall x \in V+W : \langle x, u \rangle = 0 \} \\ &= \{ u \in U \mid \forall v \in V,\ w \in W : \langle v+w, u \rangle = 0 \} \\ &= \{ u \in U \mid \forall v \in V,\ w \in W : \langle v, u \rangle = 0,\ \langle w, u \rangle = 0 \} \\ &= V^{\perp} \cap W^{\perp}.\end{aligned}$$

□

Definition 1.2. *We equip $\mathbb{k}^{s+1}$ with the bilinear form corresponding to the identity matrix. It is symmetric and nondegenerate. For a linear subvariety $L = \mathbb{P}(W)$, we call $L^{\perp} := \mathbb{P}(W^{\perp})$ the **geometric dual** of L.*

Proposition 1.3. *Let L and M be linear subvarieties of $\mathbb{P}^s$. Then,*

(a). $\dim(L^{\perp}) = s - \dim(L) - 1 = \operatorname{codim}(L) - 1.$

(b). If $L \subseteq M$, then $L^{\perp} \supseteq M^{\perp}$.

(c). $L^{\perp\perp} = L.$

(d). $(L+M)^{\perp} = L^{\perp} \cap M^{\perp}.$

Proof. Let $L = \mathbb{P}(W)$, then part (a) is the easy calculation

$$\dim(L^{\perp}) = \dim(W^{\perp}) - 1 = s - (\dim(W) - 1) - 1 = \operatorname{codim}(L) - 1$$

and the rest follows from parts (b) to (d) of Fact 1.1. □

Example 1.4. *Assume that $P, Q \in \mathbb{P}^2$ are two points in the plane. The above means that the lines $P^{\perp}$ and $Q^{\perp}$ intersect in the point dual to $P+Q$. Let us make the example more specific. Choose*

$$\begin{aligned} P &= [-1:0:1] & \qquad Q &= [-1:-1:1] \\ &= Z_*(x+z, y) & &= Z_*(x+z, y+z). \end{aligned}$$

Then, $P+Q = Z_*(x+z)$ and hence, $R := (P+Q)^{\perp} = [1:0:1]$. Furthermore,

$$P = L\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad\qquad Q = L\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Hence, $P^{\perp} = Z_*(x-z)$ and $Q^{\perp} = Z_*(x+y-z)$. In Figure 1.1, we look at the affine patch $D_*(z) \cong \mathbb{A}^2$.

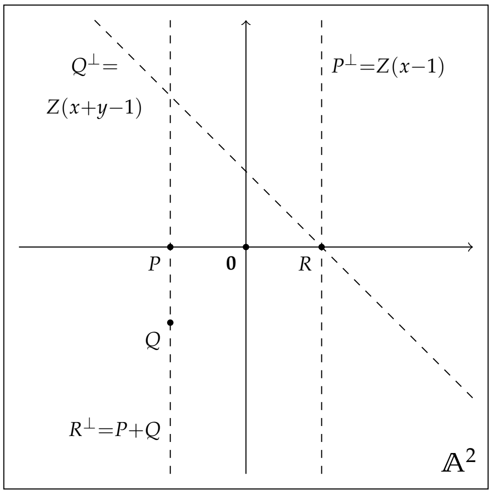

Figure 1.1: A sketch of Example 1.4.

Definition 1.5. *If $X = \{ L_1, \ldots, L_m \}$ is a set containing a finite number of linear subvarieties $L_i \subseteq \mathbb{P}^s$, we denote by $\langle X \rangle := L_1 + \cdots + L_m$, the linear span of all elements in X. Clearly,*

$$\dim \langle X \rangle \leq (m-1) + \sum_{i=1}^{m} \dim(L_i).$$

If the linear varieties intersect in a single point $L_1 \cap \cdots \cap L_m = \{ P \}$, we say that X is a ***pencil****. Furthermore, we set $X^{\perp} := \{ L_1^{\perp}, \ldots, L_m^{\perp} \}$.*

We follow [SS, Definition 3] and introduce the notion of SG_k-closedness.

Definition 1.6. *Let $X \subseteq \mathbb{P}^s$ be a finite set of points. We denote by*

$$t_k^{\perp}(d, X) := \left| \left\{ F \subseteq \mathbb{P}^s \text{ subvariety} \;\middle|\; \begin{array}{ll} \dim(F) = d, & \deg(F) = 1, \\ \langle X \cap F \rangle = F, & |X \cap F| = k. \end{array} \right\} \right|$$

the number of d-flats that intersect X in k points and are spanned by these points. Such a d-flat is said to be ***elementary*** *with respect to X if $d = k-1$.*

We say that X is SG_k***-closed*** *if it has no elementary $(k-1)$-flat. In this case, we also say that X is a* ***Sylvester-Gallai-k-Configuration***, *which we abbreviate as $\mathrm{SG}_k\mathrm{C}$. This is equivalent to saying $t_k^{\perp}(k-1, X) = 0$. We also define the number*

$$\mathrm{SG}_k(\Bbbk, m) := \max \left\{ \dim \langle X \rangle \;\middle|\; \begin{array}{l} s \in \mathbb{N},\ X \subset \mathbb{P}^s_{\Bbbk},\ |X| \leq m, \\ t_k^{\perp}(k-1, X) = 0. \end{array} \right\} + 1.$$

It is one plus the maximum dimension of a linear $\Bbbk$-variety, spanned by an $\mathrm{SG}_k\mathrm{C}$ of cardinality at most m. Note that this is the dimension of the affine cone over $\langle X \rangle$.

Example 1.7. *Let* $f := x_0^3 + x_1^3 + x_2^3 - x_0x_1x_2 \in \mathbb{C}[x_0, x_1, x_2]$. *It defines a curve* $Z_*(f) \subseteq \mathbb{P}^2_{\mathbb{C}}$, *and its inflection points are*

$$\begin{array}{lll} [0:1:-1], & [0:1:-\zeta], & [0:1:-\zeta^2], \\ [-1:0:1], & [-\zeta:0:1], & [-\zeta^2:0:1], \\ [1:-1:0], & [1:-\zeta:0], & [1:-\zeta^2:0], \end{array}$$

where $\zeta := \exp(2\pi\hat{\imath}/3)$ *is a root of unity. This can be checked easily by evaluating the determinant of the Hessian*

$$\det\left(\partial_{ij}f\right)_{i,j=0}^{2} = \det\begin{pmatrix} 6x_0 & -x_2 & -x_1 \\ -x_2 & 6x_1 & -x_0 \\ -x_1 & -x_0 & 6x_2 \end{pmatrix} = 214x_0x_1x_2 - 6x_0^3 - 6x_1^3 - 6x_2^3$$

at these points. To see that these points form an $\mathrm{SG}_2\mathrm{C}$, *simply check that any three vectors in* $\mathbb{C}^3$ *with the above coordinates are linearly dependent. In fact, the nine inflection points of any plane cubic are an* $\mathrm{SG}_2\mathrm{C}$, *see [Har, Exercise IV.2.3 (g)].*

Definition 1.8. *Dual to Definition 1.6, if* X *is a set of hyperplanes in* $\mathbb{P}^s$, *we can count the number of subspaces of codimension* d *where exactly* k *of these hyperplanes intersect. We denote this number by*

$$t_k(d, X) := \left| \left\{ F \subseteq \mathbb{P}^s \text{ subvariety} \;\middle|\; \begin{array}{l} \operatorname{codim}(F) = d, \\ |\{ H \in X \mid F \subseteq H \}| = k \\ F = \bigcap_{\substack{H \in X, \\ F \subseteq H}} H \end{array} \right\} \right|.$$

In other words, $t_k(d, X) = t_k^{\perp}(d-1, X^{\perp})$ *by geometric duality. In particular, recall Proposition 1.3.(b).*

In [SS, Theorem 4], the connection is made between Sylvester-Gallai Configurations and the rank of a depth-3 circuit. We are interested in bounding the value $\mathrm{SG}_k(\Bbbk, m)$ for $\Bbbk = \mathbb{C}$ and arbitrary k. While there is no such result known to the present date, we conjecture that

$$\forall m \in \mathbb{N} : \mathrm{SG}_k(\mathbb{C}, m) \leq 3(k-1).$$

This is equivalent to saying that for all $\mathrm{SG}_k\mathrm{C}$s X that consist of m points, there exists a subspace of dimension less or equal to $3(k-1)-1$ which completely contains X. In other words, $\exists d \leq 3(k-1) : t_m^{\perp}(d-1, X) > 0$. An arrangement H of m hyperplanes is the dual of an $\mathrm{SG}_k\mathrm{C}$ iff $t_k(k, H) = t_k^{\perp}(k-1, H^{\perp}) = 0$. In summary:

Conjecture 1.9. $\mathrm{SG}_k(\mathbb{C}, m) \leq 3(k-1)$ *for all* $m \in \mathbb{N}$. *Equivalently, if* H *is an arrangement of* m *hyperplanes in* $\mathbb{P}^s_{\mathbb{C}}$*, then:*

$$t_k(k, H) = 0 \quad \Longrightarrow \quad \exists d \leq 3(k-1) : t_m(d, H) > 0 \tag{1.1}$$

For $k \geq 3$, the issue is still an open question. For $k = 2$, we revisit the original proof of Conjecture 1.9 by Kelly, which will require a whole chapter of machinery from algebraic geometry. The proof uses Theorem 3.21 by Hirzebruch about line arrangements in the complex projective plane. Anticipating this result, we get the following:

Proposition 1.10. *If* $X \subseteq \mathbb{P}^2_{\mathbb{C}}$ *is a nonlinear* $\mathrm{SG}_2\mathrm{C}$*, then* $t_3^{\perp}(1, X) \neq 0$*. In other words, there is a line containing* exactly *three points of* X*.*

Proof. Let $X = \{ P_0, \ldots, P_\ell \}$. We consider the arrangement of lines $X^{\perp}$. Write $t_r := t_r(2, X^{\perp}) = t_r^{\perp}(1, X)$. Since X is nonlinear, $t_{\ell+1} = 0$. Since X is an $\mathrm{SG}_2\mathrm{C}$, we can also conclude $t_2 = 0$. A line L containing $P_1, \ldots, P_\ell$ must contain a point $P_i \in \ P_0 + P_1$ with $i > 2$ and thus, $P_0 \in P_0 + P_1 = P_1 + P_i = L$ implies $t_\ell = 0$. Now, Theorem 3.21 yields $t_3 \neq 0$. □

The second key to proving Conjecture 1.9 for $k = 2$ is the upcoming Proposition 1.14, which is based on [Kel, Lemma 2]. We give a more detailed proof, also for the sake of being self-contained.

Definition 1.11. *We will call* $\varphi : \mathbb{P}^s \to \mathbb{P}^s$ *a* ***linear change of coordinates*** *if* $\varphi^\sharp$ *is a degree-preserving automorphism of* $\Bbbk[x_0, \ldots, x_s]$*.*

Fact 1.12. *If* $\varphi : \mathbb{P}^s \to \mathbb{P}^s$ *is a linear change of coordinates and* $F \subseteq \mathbb{P}^s$ *a* d*-flat, then* $\varphi(F)$ *is again a* d*-flat.*

Proof. We first note that φ is a linear change of coordinates if and only if $\psi := \varphi^{-1}$ is one. If $F = Z(h_1, \ldots, h_{s-d})$ is a d-flat, then we have

$$\varphi(F) = \psi^{-1}(F) = Z(\psi^\sharp(h_1), \ldots, \psi^\sharp(h_{s-d})),$$

since $P \in \psi^{-1}(F)$ if and only if $\psi(P) \in F$, which is the case if and only if

$$\forall i : 0 = h_i(\psi(P)) = \psi^\sharp(h_i)(P).$$

Since $\psi^\sharp$ preserves degrees and linear independence, $\varphi(F)$ is a d-flat. □

Remark 1.12.1. *In particular,* $t_k^{\perp}(d, X) = t_k^{\perp}(d, \varphi(X))$ *for all* $k > 1$*. Hence, the property of being an* $\mathrm{SG}_k\mathrm{C}$ *is invariant under linear changes of coordinates.*

Fact 1.13. *If $(G, +)$ is a group and $H \subseteq G$ a finite subset which is closed under the group law, then H is a subgroup.*

Proof. We need to show that each $a \in H$ has an inverse in H. Since H is finite, there exists an $n \in \mathbb{N}$ with $n \cdot a = 0$, so $-a = (n-1) \cdot a$. $\square$

Proposition 1.14 (Kelly's Trick). *Assume that $L = \{ L_0, L_1, L_2 \}$ is a pencil of lines $L_i \subset \mathbb{P}^2_{\mathbb{k}}$ with $\dim(L_0 + L_1 + L_2) = 2$. Let $X \subseteq L_0 \cup L_1 \cup L_2$ be a finite set of points contained within the pencil. If X is a nonlinear $\mathrm{SG_2C}$, then $p := \mathrm{char}(\mathbb{k}) > 0$ and $3p \leq |X|$.*

Proof. We use $\mathbb{k}[x, y, z]$ as projective coordinates in $\mathbb{P}^2 = \mathbb{P}^2_{\mathbb{k}}$. Assume that X is an $\mathrm{SG_2C}$. We can write $L_i = Z(h_i)$ for certain linear polynomials h_i. We are going to apply a series of linear changes of coordinates until we arrive at an $\mathrm{SG_2C}$ which has the structure of an additive subgroup of $\mathbb{k}$. This is only possible if $\mathbb{k}$ has nonzero characteristic. For ease of notation, we set $X_j := X \cap L_j$. Let $P \in X$ be the point where all three lines intersect, i.e. $L_0 \cap L_1 \cap L_2 = \{P\}$.

Let $X_0 = \{ A_1, \ldots, A_q \}$ with $q > 0$. Since X is nonlinear, we may assume there is a $B \in X_1 \setminus X_0$. The line $A_i + B$ contains a third point $C_i \in X$, but since $A_i + C_i = A_i + B$, it can neither be contained in L_0 nor L_1. Thus, $A_i \mapsto C_i$ defines a bijection between X_0 and X_2. By symmetry, we conclude that $|X_j|$ does not depend on j. We denote by B_i and C_i the points of X_1 and X_2, respectively.

Since $L_0 \cap L_1 = \{ P \}$, the forms h_0 and h_1 are linearly independent – thus, there exists a linear change of variables that ensures $h_0 = y$ and $h_1 = z - y$. This immediately yields $P = [1 : 0 : 0]$. If we write

$$h_2 = \alpha x + \beta y + \gamma z,$$

then $h_2(P) = 0$ implies $\alpha = 0$. Because $L_2 \neq L_0$, we conclude $\gamma \neq 0$ and may assume $h_2 = \beta y - z$. Because $L_2 \neq L_1$, we also know that $\beta \neq 1$. For reasons that will become apparent later, we *now* assume $h_0 = (\beta - 1)y$, which changes nothing about L_0.

Let $A_i = [a_i : 0 : 1]$, $B_i = [b_i : 1 : 1]$ and $g_i = \alpha_i x + \beta_i y + \gamma_i z$ such that $A_i + B_1 = Z(g_i)$. Then, $\alpha_i \neq 0$ since otherwise, $g_i(A_i) = 0$ would imply $\gamma_i = 0$ and consequently, $g_i(B_1) = 0$ would mean $\beta_i = 0$ as well. We therefore assume

$$g_i = x + \beta_i y + \gamma_i z$$

from now on. The linear forms g_1, h_0, h_2 are linearly independent and thus,

$$\varphi := \begin{pmatrix} g_1 & \longmapsto & x \\ \beta y - y = h_0 & \longmapsto & y \\ \beta y - z = h_2 & \longmapsto & z \end{pmatrix} : \Bbbk[x,y,z] \longrightarrow \Bbbk[x,y,z]$$

defines a linear change of variables. Since $\varphi(h_1) = y - z$, we henceforth assume

$$h_0 = y, \qquad h_1 = y - z, \qquad h_2 = z \qquad \text{and} \qquad g_1 = x. \tag{1.2}$$

Note that we have maintained $P = [1 : 0 : 0]$ and we changed h_1 only by a sign, so we can still write $A_i = [a_i : 0 : 1]$ as well as $B_i = [b_i : 1 : 1]$. We note at this point that $P \notin X$ since $A_i \neq B_j$ for all i and j.

Now since $g_1 = x$, we have $a_1 = b_1 = 0$. Without loss of generality, assume $B_i = (C_1 + A_i) \cap L_2$. This implies $C_1 = (A_1 + B_1) \cap L_2 = [0 : 1 : 0]$ and consequently, $b_i = a_i$ for all i. Then,

$$\begin{aligned} g_i(a_i : 0 : 1) = 0 \;\Rightarrow\; \gamma_i = -a_i = -b_i. \\ g_i(0 : 1 : 1) = 0 \;\Rightarrow\; \beta_i = a_i = b_i. \end{aligned} \tag{1.3}$$

We now claim that $\{ a_1, \ldots, a_q \}$ defines a finite, additive subgroup of $\Bbbk$. By Fact 1.13, we only have to verify that it is closed under addition:

(a). The line $B_1 + A_i$ intersects L_2 in $C_{\tau(i)}$.

(b). The line $C_{\tau(i)} + B_j$ intersects L_0 in $A_{\sigma(i,j)}$.

(c). We claim that $a_i + a_j = a_{\sigma(i,j)}$.

Since $B_1 + A_i = Z(x + \beta_i y + \gamma_i z)$ and $L_2 = Z(z)$, we know $C_{\tau(i)} = [-\beta_i : 1 : 0]$. Let now $C_{\tau(i)} + B_j = Z(f)$ with $f = ux + vy + wz$. We may assume $u = 1$ since for $u = 0$, $f(C_{\tau(i)}) = 0$ implies $v = 0$ and then, $f(B_j)$ would mean $w = 0$. Otherwise, $v = \beta_i$ and thus, $w = -(b_j + \beta_i)$. Then, $a_{\sigma(i,j)} - b_j - \beta_i = 0$ proves our claim by (1.3).

Let $p := \mathrm{char}(\Bbbk)$. Then, $|X_0| \geq p$. This will also be true for the other two lines and $P \notin X$, so $|X| \geq 3p$. □

We remark at this point that the observation $|X| \geq 3p$ is not required in the complex case. We have included it here for Section 4.2, when we give perspectives on finite characteristic. We only need another brief lemma before we can give Kelly's proof.

Lemma 1.15. *Let $X \subseteq \mathbb{P}^s$ be SG_k-closed, $P \in X$ any point and H a hyperplane with $P \notin H$. Denote by $\pi : \mathbb{P}^s \setminus \{P\} \to H$ the linear projection from P onto H. Then, $X' := \pi(X \setminus \{P\})$ is SG_k-closed inside $H \cong \mathbb{P}^{s-1}$.*

Proof. Assume that the points $Q_1, \dots, Q_k \in X'$ span a $(k-1)$-flat $F' \subset H$. Let $P_i \in \pi^{-1}(Q_i)$ be preimages of these points and $F := P_1 + \dots + P_k$. Since $\pi(F) = F'$, we know that $\dim(F) \geq k-1$ and since F is spanned by k points, $\dim(F) = k-1$. Since $\dim(F) = \dim(F')$, we know $P \notin F$. Hence, there is a point $P_0 \in X \cap F$ which is distinct from the P_i and from P. Its image $Q_0 := \pi(P_0)$ is then contained in $F' \cap X'$. We have to show that Q_0 is distinct from the other Q_i. Hence, assume $Q_0 = Q_i$ for some $i > 0$. Then, the points P_0, P_i, Q_0 and P lie on the same line. Since $P_0 \neq P_i$, this would imply the contradiction $P \in P_0 + P_i \subseteq F$. □

Theorem 1.16. $\mathrm{SG}_2(\mathbb{C}, m) \leq 3$ *for all $m \in \mathbb{N}$.*

Proof. Let $X \subseteq \mathbb{P}^s_{\mathbb{C}}$ be an $\mathrm{SG}_2\mathrm{C}$ and assume that $\dim \langle X \rangle > 2$. Let $P \in X$ be any point and denote by $\pi : \mathbb{P}^s_{\mathbb{C}} \setminus \{P\} \to \mathbb{P}^{s-1}_{\mathbb{C}}$ the projection from P. By the above Lemma 1.15, we know that

$$Y := \pi(X \setminus \{P\})$$

is an $\mathrm{SG}_2\mathrm{C}$. By our assumption on X, we know $\dim \langle Y \rangle \geq 2$ and by induction on s, we may further assume that $\dim \langle Y \rangle \leq 2$. By Proposition 1.10, there is a line $L \subseteq \mathbb{P}^{s-1}_{\mathbb{C}}$ such that $|L \cap Y| = 3$. We now consider the intersection

$$X' := \overline{\pi^{-1}(L)} \cap X$$

of X with $\overline{\pi^{-1}(L)} \cong \mathbb{P}^2_{\mathbb{C}}$. We are left to show that it is a nonlinear $\mathrm{SG}_2\mathrm{C}$ contained in the union of three concurrent lines. This will yield the desired contradiction by Kelly's Trick (Proposition 1.14). Let $L \cap Y = \{ P_0, P_1, P_2 \}$ and $L_i := \overline{\pi^{-1}(P_i)}$. If we were to assume that there is a

$$Q \in X' \setminus (L_0 \cup L_1 \cup L_2),$$

the projection $\pi(Q)$ would be contained in

$$\pi(\pi^{-1}(L) \cap X) = L \cap Y,$$

but distinct from the P_i, which is impossible.

Thus, X' is contained in $L_0 \cup L_1 \cup L_2$. There can furthermore be no line $L' \subset \overline{\pi^{-1}(L)}$ with $|L' \cap X'| = 2$ since $L' \cap X' = L' \cap X$ and X is SG_2-closed. Thus, X' is also SG_2-closed. □

1.3 Blowing Up

The technique of "blowing up" certain parts of a variety (or scheme, if you prefer) is an essential tool in birational geometry. In fact, any birational equivalence can be understood as a blowup, see [Har, Theorem II.7.17]. We will require this tool for resolving singularities in Section 2.5. We define the notion of blowing up a variety X along an $\mathcal{O}_X$-sheaf of ideals $\mathcal{I}$. For more general introductions, see [Har, II.7] or [Liu, 8.1]. We start with the affine case:

Definition 1.17. *Let A be a ring and $I \subseteq A$ an ideal. We then let $I^0 := A$ and define a graded A-algebra*

$$A[IT] := \bigoplus_{d \geq 0} I^d T^d = \left\{ \sum_{d=0}^{n} a_d T^d \;\middle|\; \begin{array}{l} n \in \mathbb{N}, \\ \forall d : a_d \in I^d \end{array} \right\} \subseteq A[T],$$

*where T is an indeterminate. We call $A[IT]$ the **blow-up algebra of** A **in** I. If $X = \operatorname{Spec}(A)$ is an affine scheme, we call*

$$\operatorname{Bl}_I(X) := \operatorname{Proj}(A[IT])$$

*the **blow-up of** X **along** I, together with the morphism $\beta : \operatorname{Bl}_I(X) \to X$ induced by the inclusion $A \hookrightarrow A[IT]$. We refer to $Z(I)$ as the **center** of the blow-up.*

We want to give more geometric intuition to this purely algebraic definition. First, we need to recall one basic notion: If $\varphi : X \to Y$ is a morphism of varieties, then

$$\begin{tikzcd}
X \arrow[dr, dotted, "\mathrm{id}_X \times \varphi"] \arrow[drr, bend left, "\varphi"] \arrow[ddr, bend right, "\mathrm{id}_X"'] & & \\
& X \times Y \arrow[r] \arrow[d] & Y \arrow[d] \\
& X \arrow[r] & \operatorname{Spec}(\Bbbk)
\end{tikzcd}$$

$\mathrm{id}_X \times \varphi$ is a closed immersion whose image $\Gamma(\varphi) := \operatorname{im}(\mathrm{id}_X \times \varphi)$ we call the **graph of** φ. Its closed points are just equal to

$$\Gamma(\varphi)(\Bbbk) = \{\, (x, \varphi(x)) \mid x \in X \,\} \subseteq X \times Y.$$

We can now easily extend this definition to rational maps:

Definition 1.18. *If $\varphi : X \dashrightarrow Y$ is a rational map defined on the open subset $U \subseteq X$, we denote by $\Gamma(\varphi) \subseteq X \times Y$ the closure of the graph of the regular map $\varphi_U : U \to Y$ and call it the **graph of** φ.*

Proposition 1.19. *Let $X = \operatorname{Spec}(A)$ be an affine variety and $I \subseteq A$ a nonzero ideal. Let $Y := Z(I)$ and pick generators $I = (f_0, \dots, f_r)$. We define a rational map $\varphi : X \dashrightarrow \mathbb{P}^r$ on the closed points of $U := X \setminus Y$ by*

$$\varphi(P) := [f_0(P) : \dots : f_r(P)].$$

In other words, φ is induced by the line bundle $I^{\sim}$. Then, $\mathrm{Bl}_I(X) \cong \Gamma(\varphi)$ is a quasi-projective variety and β corresponds to $\Gamma(\varphi) \hookrightarrow X \times \mathbb{P}^r \twoheadrightarrow X$ under this identification.

Proof. There is a surjection of graded $\Bbbk$-algebras

$$\begin{aligned} \pi : A[y_0, \dots, y_r] &\longrightarrow A[IT] \\ y_i &\longmapsto f_i T, \end{aligned}$$

corresponding to a closed embedding $\iota : \mathrm{Bl}_I(X) \hookrightarrow X \times \mathbb{P}^r$. Since obviously

$$\left(f_i y_j - f_j y_i \mid 0 \le i, j \le r \right) \subseteq \ker(\pi),$$

we can see that $\mathrm{Bl}_I(X) \subseteq \Gamma(\varphi)$. Since $\dim(A[IT]) > \dim(A)$, we also know

$$\begin{aligned} \dim(\mathrm{Bl}_I(X)) = \dim(\operatorname{Proj}(A[IT])) &= \dim(A[IT]) - 1 \ge \dim(A) \\ &= \dim(X) = \dim(\Gamma(\varphi)), \end{aligned}$$

implying $\dim(\mathrm{Bl}_I(X)) = \dim(\Gamma(\varphi))$. The result follows because both varieties are irreducible and closed. $\square$

Corollary 1.20. *With notation as in Proposition 1.19, let $V := \beta^{-1}(U)$. Then,*

$$\beta|_V : V \xrightarrow{\sim} U = X \setminus Y$$

is an isomorphism of varieties. Thus, β is a birational equivalence and in particular, $\dim(\mathrm{Bl}_I(X)) = \dim(X)$. $\square$

Proposition/Definition 1.21. *Let $\beta : \mathrm{Bl}_I(X) \to X$ be the blow-up of an affine variety $X = \operatorname{Spec}(A)$ along some ideal I. The homogeneous ideal*

$$I \cdot A[IT] = \bigoplus\nolimits_{d \ge 0} I^{d+1} T^d$$

*is called the **exceptional ideal** of the blow-up. Any localization of it by an element in degree one is a principal ideal and the associated Cartier divisor E is called the **exceptional divisor**. Let Y be the center of β, then E is supported on $\beta^{-1}(Y)$.*

Proof. Let $f \in I$. In $(A[IT]_{fT})_0$, we have $(gT/fT) \cdot f = g$ for every $g \in I$, so $(I_{fT})_0 = (f)$ is principal. For any homogeneous prime ideal $P \subset A[IT]$, the inclusion $I \subseteq A \cap P$ holds if and only if $I \cdot A[IT] \subseteq P$. In other words, $\beta(P) \in Y$ if and only if $P \in E$. This means $\beta^{-1}(Y) = E$. $\square$

Notation 1.22. *If $I(Y) = I$ for a closed subscheme $Y \subseteq X$, we write $\mathrm{Bl}_Y(X)$ instead of $\mathrm{Bl}_I(X)$ and call it the **blow-up of** X **along** Y. We sometimes also write $\mathrm{Bl}(X, Y)$ instead of $\mathrm{Bl}_Y(X)$.*

We now generalize to arbitrary schemes. In the following, $\mathbb{P}\mathbf{roj}$ denotes the *relative proj-construction*, as explained very comprehensively in [Liu, Chapter 8.1, Lemma 8.1.8] and also in [Har, II.7].

Definition 1.23. *Let X be a Noetherian scheme and $\mathcal{I}$ an $\mathcal{O}_X$-sheaf of ideals. We define the sheaf of graded algebras*

$$\mathcal{O}_X[\mathcal{I}T] := \bigoplus_{d \geq 0} \mathcal{I}^d T^d \subseteq \mathcal{O}_X[T]$$

*where $\mathcal{I}^0 := \mathcal{O}_X$. The **blow-up of** X **along** $\mathcal{I}$ is then defined as*

$$\mathrm{Bl}_{\mathcal{I}}(X) := \mathbb{P}\mathbf{roj}(\mathcal{O}_X[\mathcal{I}T]).$$

*The closed subscheme $\mathcal{Z}(\mathcal{I})$ is called the **center** of the blow-up. As in Notation 1.22, we set $\mathrm{Bl}_Y(X) := \mathrm{Bl}_{\mathcal{I}(Y)}(X)$ for closed subschemes $Y \hookrightarrow X$.*

Consider now a closed subvariety Z of X passing through the center Y of a blow-up. Its preimage will contain the exceptional divisor, but it will have a second component $\tilde{Z}$, which is the same as Z, outside of Y. It is called the *strict transform* of Z. To properly define and study it, we first ponder on some less elementary properties of the blow-up such as functoriality and its universal property.

Definition 1.24. *If $\varphi : Y \to X$ is a morphism of schemes and $\mathcal{I}$ is an $\mathcal{O}_X$-sheaf of ideals, consider the exact sequence $\mathbf{0} \to \mathcal{I} \hookrightarrow \mathcal{O}_X$. Since the pull-back is in general not left-exact, the map $\alpha : \varphi^*(\mathcal{I}) \to \varphi^*(\mathcal{O}_X) \cong \mathcal{O}_Y$ might not be a monomorphism. We call $\varphi^\star(\mathcal{I}) := \mathrm{im}(\alpha)$ the **inverse image ideal sheaf of** $\mathcal{I}$ **under** φ.*

Fact 1.25. *In terms of Definition 1.24, $\varphi^\star(\mathcal{I}) \cong \varphi^{-1}(\mathcal{I}) \cdot \mathcal{O}_Y$.*

Proof. Let $U = \mathrm{Spec}(A) \subseteq X$ and $V = \mathrm{Spec}(B) \subseteq \varphi^{-1}(U)$. With $I := \mathcal{I}(V)$,

$$\begin{tikzcd} \varphi^*(\mathcal{I})(V) \arrow[r, equal] \arrow[d, dashed, "\alpha_V"] & B \otimes_A I \arrow[d] \\ \varphi^*(\mathcal{O}_Y)(V) \arrow[r, equal] & B \end{tikzcd}$$

so $\alpha_V(b \otimes t) = bt$ and $\mathrm{im}(\alpha_V) = IB = (\varphi^{-1}(\mathcal{I}) \cdot \mathcal{O}_Y)(V)$. This induces local isomorphisms $\mathrm{im}(\alpha)|_V \cong (\varphi^{-1}(\mathcal{I}) \cdot \mathcal{O}_Y)|_V$ of $\mathcal{O}_V$-modules which agree on stalks and therefore glue. $\square$

Fact 1.26. *Let $\mathcal{I}$ be an invertible sheaf of ideals on a $\Bbbk$-variety X. Then, the blow-up $\beta : \mathrm{Bl}_{\mathcal{I}}(X) \to X$ is an isomorphism.*

Proof. We may harmlessly assume that $X = \mathrm{Spec}(A)$ is affine and $\mathcal{I}$ corresponds to a principal ideal $(f) \subseteq A$. Hence,

$$\mathrm{Bl}_I(X) = \mathrm{Proj}(A[fT]) \cong \mathrm{Proj}(A[T]) \cong \mathrm{Spec}(A) = X. \qquad \square$$

Theorem 1.27 (Functoriality of the Blow-Up). *Let $\varphi : X \to X'$ be a morphism of $\Bbbk$-varieties and $\mathcal{J}' \subseteq \mathcal{O}_{X'}$ an ideal sheaf such that the inverse image $\mathcal{J} := \varphi^\star(\mathcal{J}')$ is invertible. Then,*

$$\begin{tikzcd} \mathrm{Bl}_{\mathcal{J}}(X) \arrow[r, "\beta"] \arrow[d, dotted, "\exists! \bar{\varphi}"'] \arrow[dr, phantom, "\circlearrowleft"] & X \arrow[d, "\varphi"] \\ \mathrm{Bl}_{\mathcal{J}'}(X') \arrow[r, "\beta'"'] & X' \end{tikzcd} \tag{1.4}$$

This construction is functorial and preserves closed embeddings.

Remark. This is [Har, Corollary II.7.15] for varieties, but we give a proof here that uses our characterization from Proposition 1.19 rather than the universal property described in [Har, Proposition II.7.14]. Instead, we will use functoriality to deduce the universal property next.

Proof. Since the blow-up is local and in view of the asserted uniqueness of $\bar{\varphi}$, we may assume that $X = \mathrm{Spec}(A)$ and $X' = \mathrm{Spec}(A')$ are both affine varieties. Let $I' := \mathcal{J}'(X') = (f_0', \dots, f_r')$, so $I := \mathcal{J}(X) = (f_0, \dots, f_r)$ for $f_i := \varphi^\sharp(f_i')$. Since we assumed $\mathcal{J}$ to be an invertible sheaf, $I \neq (0)$. The induced morphisms $\psi : X \dashrightarrow \mathbb{P}^r$ and $\psi' : X' \dashrightarrow \mathbb{P}^r$ satisfy $\psi = \psi' \circ \varphi$ since $f_i = f_i' \circ \varphi$ as regular maps. We write $U := X \setminus Z(I)$ and $U' := X' \setminus Z(I')$, then there is a unique morphism

$$\begin{tikzcd} V := \Gamma(\psi_U) \arrow[r, two heads] \arrow[d, dotted, "\bar{\varphi}_V"'] & U \arrow[d, "\varphi|_U"] \\ V' := \Gamma(\psi'_{U'}) \arrow[r, two heads] & U' \end{tikzcd}$$

which maps $\bar{\varphi}_V(P, \psi(P)) := (\varphi(P), \psi(P)) = (\varphi(P), \psi'(\varphi(P)))$. Furthermore, there certainly exist graded maps of A'-algebras $\bar{\varphi}^\sharp$ that make the diagram

$$\begin{tikzcd} A[IT] & A \arrow[l, hook'] \\ A'[I'T'] \arrow[u, dotted, "\bar{\varphi}^\sharp"] & A' \arrow[l, hook'] \arrow[u, "\varphi^\sharp"'] \end{tikzcd}$$

commute: For instance, take the map defined by $T' \mapsto T$. The induced morphisms $\bar{\varphi} : \mathrm{Bl}_I(X) \to \mathrm{Bl}_{I'}(X')$ satisfy $\bar{\varphi}|_V = \bar{\varphi}_V$, so they all agree on a dense open subset of $\mathrm{Bl}_I(X)$ and must therefore be equal.

The morphism φ is a closed immersion if and only if $\varphi^\sharp$ is surjective. In this case, $\bar{\varphi}^\sharp$ is also surjective and $\bar{\varphi}$ a closed immersion. □

Corollary 1.28 (Universal Property Of Blowing Up). *Let $\varphi : Y \to X$ be a morphism of varieties and $\mathcal{I}$ a coherent sheaf of ideals on X. Let $\beta : \tilde{X} := \mathrm{Bl}_{\mathcal{I}}(X) \to X$. If $\varphi^\star(\mathcal{I})$ is an invertible sheaf on Y, then*

$$\begin{tikzcd} Y \arrow[r, dotted, "\exists ! \bar{\varphi}"] \arrow[dr, "\varphi"'] & \tilde{X} \arrow[d, "\beta"] \arrow[dl, phantom, "\circlearrowleft"] \\ & X \end{tikzcd}$$

there exists a unique $\bar{\varphi} : Y \to \tilde{X}$ with $\beta \circ \bar{\varphi} = \varphi$.

Proof. Since $\mathcal{J} := \varphi^\star(\mathcal{I})$ is invertible, the blow-up $\alpha : \tilde{Y} := \mathrm{Bl}_{\mathcal{J}}(Y) \to Y$ is an isomorphism by Fact 1.26. Hence, we are done by Theorem 1.27. □

Definition 1.29. *Assume that $\imath : Z \hookrightarrow X$ is a closed immersion of $\Bbbk$-varieties and $\beta : \mathrm{Bl}_{\mathcal{I}}(X) \to X$ a blow-up of X. Then, we set $\tilde{\mathcal{I}} := \imath^\star(\mathcal{I})$ and define the **strict transform of** Z to be $\beta^\intercal(Z) := \mathrm{im}(\tilde{\imath})$, where $\tilde{\imath}$ is the induced morphism*

$$\begin{tikzcd} \mathrm{Bl}_{\tilde{\mathcal{I}}}(Z) =: \tilde{Z} \arrow[r, "\gamma"] \arrow[d, hook, "\tilde{\imath}"] \arrow[dr, phantom, "\circlearrowleft"] & Z \arrow[d, hook, "\imath"] \\ \mathrm{Bl}_{\mathcal{I}}(X) =: \tilde{X} \arrow[r, "\beta"] & X \end{tikzcd} \tag{1.5}$$

Proposition 1.30. *With notation as in Definition 1.29, let $\mathcal{J} := \mathcal{I}(Z)$. Then, the ideal corresponding to $\tilde{Z} = \beta^\intercal(Z)$ is equal to*

$$\bigoplus_{d \geq 0} (\mathcal{I}^d \cap \mathcal{J}) \cdot T^d \tag{1.6}$$

Proof. First note that for closed embeddings $\imath : Z \hookrightarrow X$, the pull-back is an exact functor, so $\imath^*(\mathcal{I}) = \imath^\star(\mathcal{I})$ is just the pullback of $\mathcal{I}$. Since we are dealing with quasi-coherent sheaves, we may assume that $X = \mathrm{Spec}(A)$ is affine and the closed immersion $\imath$ of $Z = \mathrm{Spec}(A/J)$ into X is given by the surjection of rings $\imath^\sharp : A \twoheadrightarrow A/J$. Then by definition,

$$\tilde{Z} = \mathrm{Proj}\left(\bigoplus\nolimits_{d \geq 0} \left(I^d \cdot A/J\right) \cdot T^d\right).$$

The induced closed immersion $\tilde{\imath} : \tilde{Z} \hookrightarrow \tilde{X}$ corresponds to a surjection of graded rings

$$\tilde{\imath}^\sharp : \bigoplus_{d\geq 0} I^d \cdot T^d \twoheadrightarrow \bigoplus_{d\geq 0} \left(I^d \cdot A/J\right) \cdot T^d$$

whose kernel is clearly equal to (1.6) □

Proposition 1.31. *With notation as in Definition 1.29 and $Y := \mathcal{Z}(\mathcal{I})$,*

$$\beta^\intercal(Z) = \overline{\beta^{-1}(Z \setminus Y)}.$$

Proof. Let $E := \beta^{-1}(Y)$ and $\tilde{E} := \alpha^{-1}(Y \cap Z) = E \cap \tilde{Z}$. Since horizontal morphisms in (1.5) become isomorphisms when restricting to the open subset $U := X \setminus Y$, we know $\tilde{\imath}(\tilde{Z}) \setminus E = \beta^{-1}(Z \setminus Y)$. Since $\tilde{\imath}(\tilde{E}) \subseteq E$,

$$\overline{\tilde{\imath}(\tilde{Z}) \setminus E} = \overline{\tilde{\imath}(\tilde{Z}) \setminus \tilde{\imath}(\tilde{E})} = \overline{\tilde{\imath}(\tilde{Z} \setminus \tilde{E})} = \tilde{\imath}\left(\overline{\tilde{Z} \setminus \tilde{E}}\right) = \tilde{\imath}(\tilde{Z}).$$

□

Corollary 1.32. *We consider closed subvarieties $Z_1, \ldots, Z_r \subseteq X$ of a variety X under the blowing-up $\beta : \mathrm{Bl}_{\mathcal{I}}(X) \to X$. Let $Y := \mathcal{Z}(\mathcal{I})$.*

(a). $\beta^\intercal(\prod_{i=1}^r Z_i) = \prod_{i=1}^r \beta^\intercal(Z_i)$.

(b). If $Y \supseteq \prod_{i=1}^r Z_i$, then $\prod_{i=1}^r \beta^\intercal(Z_i) = \emptyset$.

(c). $\beta^\intercal(\bigcup_{i=1}^r Z_i) = \bigcup_{i=1}^r \beta^\intercal(Z_i)$.

(d). If Z_i is irreducible, then so is $\beta^\intercal(Z_i)$.

Proof. Parts (a) and (b) are the result of Proposition 1.30, since

$$\bigoplus_{d\geq 0}\left(\left(\sum_{i=1}^r \mathcal{J}_i\right) \cap \mathcal{I}^d\right) T^d = \sum_{i=1}^r \left(\bigoplus_{d\geq 0} \left(\mathcal{J}_i \cap \mathcal{I}^d\right) T^d\right)$$

and the strict transform of Y is clearly empty.

Parts (c) and (d) follow directly from Proposition 1.31 since

$$\overline{\beta^{-1}\left(\bigcup_{i=1}^r Z_i \setminus Y\right)} = \bigcup_{i=1}^r \overline{\beta^{-1}(Z_i \setminus Y)}.$$

and if Z_i is irreducible, then $Z_i \setminus Y$ is an irreducible, closed subset of the open set $U := X \setminus Y$. Hence, $\overline{\beta^{-1}(Z_i \setminus Y)}$ is also irreducible. □

A self-contained proof of the following well-known result would require more commutative algebra than the scope of our introduction permits.

Theorem 1.33. *Let X be a nonsingular $\Bbbk$-variety and $Y \subseteq X$ a nonsingular, closed subvariety. Then, both $\mathrm{Bl}_Y(X)$ and the exceptional divisor of this blow-up are nonsingular $\Bbbk$-varieties.*

Metaproof. See [Har, Theorem II.8.24]. □

1.4 Intersection Theory and Chern Classes

An *intersection theory* should make it possible to calculate intersections of subvarieties, counted with "multiplicities". We can only give a very brief overview of the basic terminology for this rather vast area of study. For a detailed introduction, see [Ful1]. At the time of writing, the author would also recommend the excellent lecture notes [Gat, Chapters 9 and 10]. For brevity, however, we follow the axiomatic approach of [Har, Appendix A] and assume $\Bbbk$ to be an algebraically closed field.

Definition 1.34. *Let X be an s-dimensional variety over $\Bbbk$. Let $Z^k(X)$ be the free abelian group generated by all subvarieties $Y \subseteq X$ of codimension k and define the graded group $Z(X) := \bigoplus_{k=0}^{s} Z^k(X)$. An element of $Z(X)$ is called a* ***cycle****. A cycle is* ***positive*** *if each of its coefficients is a positive integer number.*

To be able to count intersections with multiplicities, we need to be able to "move" varieties around without changing the result of their intersection. The correct notion for this is *rational equivalence*.

Definition 1.35. *If M is an A-module, we denote by $\operatorname{len}_A(M)$ the* ***length*** *of M over A. It is the supremum of all lengths r of chains $\mathbf{0} \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots \subsetneq M_r = M$ of submodules $M_i \subseteq M$. We write $\operatorname{len}(A)$ to denote the length of A as an A-module.*

Definition 1.36. *Let X be an s-dimensional $\Bbbk$-variety. If $Y \subseteq X$ is a closed subvariety and $f \in \Bbbk(Y)$, we set*

$$\operatorname{div}(f) := \sum_{\operatorname{codim}_Y(Z)=1} \operatorname{ord}_Z(f) \cdot Z.$$

Recall that the ***order*** *of an element $f \in \mathcal{O}_{Y,Z}$ is defined to be*

$$\operatorname{ord}_Z(f) := \operatorname{len}_{\mathcal{O}_{Y,Z}} \left(\mathcal{O}_{Y,Z}/(f)\right).$$

We then extend this definition to the function field $\Bbbk(Y) = \operatorname{Frac}(\mathcal{O}_{Y,Z})$ by requiring that $\operatorname{ord}(f/g) = \operatorname{ord}(f) - \operatorname{ord}(g)$.

A cycle which is of the form $\operatorname{div}(f)$ is called ***rational****. The free abelian subgroup of $Z^k(X)$, generated by all rational cycles, is denoted $\operatorname{Rat}^k(X)$. For $W, V \in Z^k(X)$, we write $W \sim V$ if $W - V \in \operatorname{Rat}^k(X)$. We say that V and W are* ***rationally equivalent*** *in this case. The* ***Chow ring of*** *X is the graded ring $A(X) = \bigoplus_{k=0}^{s} A^k(X)$ where $A^k(X)$ is the factor group*

$$A^k(X) := Z^k(X) \Big/ \operatorname{Rat}^k(X) \,.$$

The elements of $A(X)$ are called ***cycle classes****. A cycle class is* ***positive*** *if it can be represented by a positive cycle. We write $[Y]$ for the equivalence class of Y.*

A cycle class can now be "moved" along rational cycles. Note that this is a generalization of the linear equivalence between the divisors $\mathrm{Div}(X) = Z^1(X)$. Hence, $A^1(X) = \mathrm{Pic}(X)$. One then proceeds to construct an **intersection product**

$$\begin{aligned} A^k(X) \times A^j(X) &\longrightarrow A^{k+j}(X) \\ [Y], [Z] &\longmapsto [Y] \cdot [Z] \end{aligned} \tag{1.7}$$

for each variety X. Unfortunately, it would be outside the scope of this thesis to explain the construction in detail.

Definition 1.37. *Let $\varphi : X \to X'$ be a morphism of varieties and $Y \subseteq X$ a closed subvariety. If $\dim(\varphi(Y)) < \dim(Y)$, we set $\varphi_*([Y]) := 0$. Otherwise, $\Bbbk(Y)$ is a finite extension field of $\Bbbk(Y')$, where $Y' = \overline{\varphi(Y)}$. We then set*

$$\varphi_*([Y]) := [\Bbbk(Y) : \Bbbk(Y')] \cdot [Y'].$$

On the other hand, if $Y' \subseteq X'$ is any closed subvariety, denote by $\Gamma(\varphi) \subseteq X \times X'$ the graph of φ and set

$$\varphi^*([Y']) := p_*\Big([\Gamma(\varphi)] \cdot [q^{-1}(Y')]\Big).$$

Here, p and q are the projections from $X \times X'$ to X and X', respectively.

One can then show that (1.7) has the following properties:

A1. The pairing (1.7) turns $A(X)$ into a commutative, graded, unitary ring for every variety X.

A2. For $\varphi : X \to X'$, the pull-back $\varphi^* : A(X') \to A(X)$ is a morphism of graded rings. Also, $\varphi^* \circ \psi^* = (\psi \circ \varphi)^*$ for $\psi : X' \to X''$.

A3. For a proper $\varphi : X \to X'$, the push-forward $f_* : A(X) \to A(X')$ is a morphism of graded groups. Also, $\psi_* \circ \varphi_* = (\psi \circ \varphi)_*$ if $\psi : X' \to X''$.

A4. For $[Y] \in A(X)$ and $[Y'] \in A(X')$, $\varphi_*([Y] \cdot \varphi^*([Y'])) = \varphi_*([Y]) \cdot Y'$.

A5. For $[Y], [Z] \in A(X)$, $[Y] \cdot [Z] = \delta^*([Y \times Z])$, where $\delta : X \to X \times X$ is the diagonal morphism.

A6. Let Y and Z be subvarieties of X and let $Y \cap Z = W_1 \cup \cdots \cup W_r$ be the irreducible components of their intersection. Assume that Z and Y **intersect properly**, i.e. $\mathrm{codim}_X(W_i) = \mathrm{codim}_X(Y) + \mathrm{codim}_X(Z)$ for all i. Then, there exist intersection multiplicities $\mu_j \in \mathbb{Z}$ such that

$$[Y] \cdot [Z] = \textstyle\sum_{j=1}^r \mu_j \cdot [W_j].$$

The number μ_j can be calculated as follows: Let $R := \mathcal{O}_{X,W_j}$ be the local ring at W_j and let I and J denote the ideals of R that correspond to Y and Z, respectively. Then,

$$\mu_j = \sum_{k \in \mathbb{N}} (-1)^k \cdot \operatorname{len}_R \left(\operatorname{Tor}_k^R \left(R/I \, , \, R/J \right) \right).$$

where $\operatorname{Tor}_k^R(-, M)$ denotes the k-th left derived functor of the tensor product functor $(-) \otimes_R M$.

A7. If Y is a subvariety of X and Z is an effective Cartier divisor meeting Y propertly, then $[Y] \cdot [Z]$ is the cycle class associated to the cartier divisor $Y \cap Z$ on Y, which is defined by restricting the local equation of Z to Y.

In particular, that the transversal intersection of nonsingular subvarieties have multiplicity one.

In fact, properties A1 to A7 uniquely characterize the intersection product, see [Har, Theorem A.1.1]. There are two more properties of the intersection product that can be deduced from the above:

A8. For any affine space $\mathbb{A}^s$, the projection $p : X \times \mathbb{A}^s \to X$ induces an isomorphism $p^* : A(X) \xrightarrow{\sim} A(X \times \mathbb{A}^s)$.

A9. If Y is a nonsingular, closed subvariety of X and $U = X \setminus Y$ its complement, there is an exact sequence

$$A(Y) \overset{\jmath_*}{\hookrightarrow} A(X) \overset{\imath^*}{\twoheadrightarrow} A(U) \longrightarrow \mathbf{0}$$

where $\jmath : Y \hookrightarrow X$ and $\imath : U \hookrightarrow X$ are the inclusion morphisms.

Example 1.38. *$A(\mathbb{P}^s) \cong \mathbb{Z}[h]/\left(h^{s+1}\right)$, where h in degree 1 is the class of a hyperplane. We prove this by induction on s. For $s = 0$, the statement is obvious. Otherwise, pick two hyperplanes H and H' that meet transversally, so $h = [H] = [H']$ and $g := [H \cap H']$. Set $U := \mathbb{P}^s \setminus H \cong \mathbb{A}^s$ in property A9 and note that $H \cong \mathbb{P}^{s-1}$. Then, by induction and property A8, we have a sequence*

$$\mathbb{Z}[g]/(g^s) \xrightarrow{\jmath_*} A(\mathbb{P}^s) \overset{\imath^*}{\twoheadrightarrow} A(\mathbb{A}^s) \cong \mathbb{Z} \longrightarrow \mathbf{0}$$

For certain generic, transversal hyperplanes H_i,

$$\begin{aligned} g^{k-1} &= [(H \cap H_1) \cap (H \cap H_2) \cap \cdots \cap (H \cap H_{k-1})] \\ &= [H_1 \cap \cdots \cap H_{k-1} \cap H] = h^k, \end{aligned}$$

so $\jmath_*(g^{k-1}) = h^k$ is a generator in degree $k > 0$. This also shows that $\jmath_*$ is injective.

Cycle classes in degree zero can now be understood as closed points, counted with multiplicity. We use the notation of [Ful1] for counting them:

Definition 1.39. *Let X be a variety and $\alpha \in A(X)$ a cycle class whose degree-zero part can be written as $\alpha_0 = \sum_{i=1}^N n_i[P_i]$ for certain points $P_i \in X$. Then, we define the **degree of** α as*

$$\int_X \alpha := \sum_{i=1}^N n_i.$$

Hence, if $\pi_X : X \to \mathrm{Spec}(\Bbbk)$ is the structure morphism,

$$\int_X \alpha = \pi_{X*}(\alpha),$$

where we implicitly use the canonical isomorphism $A(\mathrm{Spec}(\Bbbk)) \cong \mathbb{Z}$. Hence, for any proper $\psi : X \to X'$, we have

$$\int_{X'} \psi_*(\alpha) = \pi_{X'*}(\psi_*(\alpha)) = (\pi_{X'} \circ \psi)_*(\alpha) = \pi_{X*}(\alpha) = \int_X \alpha.$$

To define Chern classes, we now need a generalization of Example 1.38, which we will state without proof. Recall that the **symmetric algebra** of a sheaf $\mathscr{E}$ of $\mathcal{O}_X$-modules is the sheaf $\mathrm{Sym}(\mathscr{E})$ associated to the presheaf

$$U \longmapsto \bigoplus_{d\geq 0} \mathscr{E}(U)^{\otimes d} \Big/ (\, f \otimes g - g \otimes f \mid f, g \in \mathscr{E}(U)\,)\,,$$

see also [Har, Exercise II.5.16].

Lemma 1.40. *Let $\mathscr{E}$ be a locally free sheaf of rank r on a variety X over $\Bbbk$. Let*

$$\pi : \mathbb{P}(\mathscr{E}) = \mathbb{P}\mathbf{roj}(\mathrm{Sym}(\mathscr{E})) \longrightarrow X$$

be the associated projective space bundle and let $h \in A^1(\mathbb{P}(\mathscr{E}))$ be the class of the divisor corresponding to $\mathcal{O}_{\mathbb{P}(\mathscr{E})}(1)$. Then, $A(\mathbb{P}(\mathscr{E}))$ is a free $A(X)$-module via π^*, generated by h^k for $0 \leq k \leq r-1$.

Definition 1.41. *Let $\mathscr{E}$ be a locally free sheaf of rank r on a nonsingular, quasi-projective variety X over $\Bbbk$. Using the notation and statement of Lemma 1.40, we can write*

$$-h^r = \sum_{k=1}^r (-1)^k \cdot \pi^*(c_k) \cdot h^{r-k}$$

*We then define the **k-th Chern class of** $\mathscr{E}$ to be $c_k(\mathscr{E}) := c_k \in A^k(X)$. We also set $c_0(\mathscr{E}) := c_0 := 1$, so $\sum_{k=0}^r (-1)^k \cdot \pi^*(c_k) \cdot h^{r-k} = 0$. The **total Chern class** is the sum $c(\mathscr{E}) := \sum_{k=0}^r c_k(\mathscr{E})$ and for a formal variable T, we define the **Chern polynomial***

$$c_T(\mathscr{E}) := \sum_{k=0}^r c_k(\mathscr{E}) \cdot T^k.$$

While this definition is very formal, it can be shown that the Chern classes of a variety are subject to several useful properties:

C1. If $\mathscr{E}$ is a line bundle corresponding to a divisor class $[D] \in A^1(X)$, then $c_T(\mathscr{E}) = 1 + [D] \cdot T$. Indeed, in this case, $\mathbb{P}(\mathscr{E}) = X$ and $\mathcal{O}_{\mathbb{P}(\mathscr{E})}(1) = \mathscr{E}$, so $h = [D]$ in Lemma 1.40. Hence, by definition, $c_0(\mathscr{E}) \cdot [D] - c_1(\mathscr{E}) = 0$.

C2. If $\varphi : X' \to X$ is a morphism and $\mathscr{E}$ is a locally free sheaf on X, then $c_k(\varphi^*\mathscr{E}) = \varphi^*(c_k(\mathscr{E}))$ for each k.

C3. If $0 \to \mathscr{E}' \hookrightarrow \mathscr{E} \twoheadrightarrow \mathscr{E}'' \to 0$ is an exact sequence of locally free sheaves on X, then $c_T(\mathscr{E}) = c_T(\mathscr{E}') \cdot c_T(\mathscr{E}'')$.

Again, one can show that these already uniquely define a theory of Chern classes, which assigns to each locally free sheaf $\mathscr{E}$ on some variety X an element $c_k(\mathscr{E}) \in A^k(X)$ and satisfies properties C1 to C3. For the proof of this, one requires the following

Theorem 1.42 (Splitting Principle). *Let $\mathscr{E}'$ be a locally free sheaf on a variety X'. Then, there exists a morphism $\varphi : X \to X'$ such that $\varphi^* : A(X') \hookrightarrow A(X)$ is injective and $\mathscr{E} := \varphi^*(\mathscr{E}')$* ***splits****, i.e. has a filtration*

$$\mathscr{E} = \mathscr{E}_0 \supseteq \mathscr{E}_1 \supseteq \cdots \supseteq \mathscr{E}_r = \mathbf{0}$$

whose successive quotients $\mathscr{L}_k := \mathscr{E}_k / \mathscr{E}_{k-1}$ are invertible sheaves.

Then, one deduces the following property C4 from property C3. The uniqueness is then a result of property C1.

C4. If $\mathscr{E}$ splits and the filtration has the invertible sheaves $\mathscr{L}_1, \ldots, \mathscr{L}_r$ as quotients, then $c_T(\mathscr{E}) = \prod_{k=1}^r c_T(\mathscr{L}_k)$.

C5. Let us write

$$c_T(\mathscr{E}) = \prod_{i=1}^r (1 + a_i T) \qquad\qquad c_T(\mathscr{F}) = \prod_{j=1}^s (1 + b_j T)$$

for two locally free sheaves $\mathscr{E}$ and $\mathscr{F}$ on X, where the a_k and b_k are just formal symbols. Then,

$$\begin{aligned} c_T(\mathscr{E}^\vee) &= \textstyle\prod_{i=1}^r (1 - a_i T), \\ c_T\left(\bigwedge^p \mathscr{E}\right) &= \textstyle\prod_{\substack{\lambda \subset \{1,\ldots,r\} \\ |\lambda| = p}} \left(1 + \sum\limits_{i \in \lambda} a_i T\right), \\ c_T(\mathscr{E} \otimes \mathscr{F}) &= \textstyle\prod_{i,j} (1 + (a_i + b_j)T). \end{aligned}$$

Remark 1.43. *Note that the expressions in property C5 make sense: When multiplied out, the coefficients of each power of T are symmetric functions in the a_i and b_j. By a well-known theorem on symmetric functions, they can be expressed as polynomials in the elementary symmetric funtions of the a_i and the b_j which are none other than the Chern classes of $\mathscr{E}$ and $\mathscr{F}$.*

In the context of the Hirzebruch-Riemann-Roch theorem, the formal calculus of Chern classes is extended by the notions of exponential Chern character and Todd class:

Definition 1.44. *Let $\mathscr{E}$ be a locally free sheaf of rank r on a variety X over $\Bbbk$ and write $c_T(\mathscr{E}) = \prod_{i=1}^{r}(1 + a_i T)$ with formal variables a_i. The **exponential Chern character** is defined to be*

$$\mathrm{ch}(\mathscr{E}) := \sum_{i=1}^{r} \exp(a_i)$$

*where we formally set $\exp(a) := \sum_{k=0}^{\infty} \frac{a^k}{k!}$. Furthermore, the **Todd class** of $\mathscr{E}$ is the formal expression*

$$\mathrm{td}(\mathscr{E}) := \prod_{i=1}^{r} \frac{a_i}{1 - \exp(-a_i)}.$$

We recall the following

Definition 1.45. *If $\mathscr{E}$ is a sheaf of $\mathcal{O}_X$-modules, then*

$$\chi(X, \mathscr{E}) := \sum_{k \in \mathbb{Z}} (-1)^k \cdot \mathrm{rank}\left(\mathcal{H}^k(X, \mathscr{E})\right)$$

is the Euler characteristic of $\mathscr{E}$.

Now, we have all the vocabulary at hand to quote the famous result which was proved by Hirzebruch over $\mathbb{C}$ and later generalized to any algebraically closed field $\Bbbk$ by Borel and Serre.

Theorem 1.46 (The Hirzebruch-Riemann-Roch Theorem). *For any locally free sheaf $\mathscr{E}$ on a nonsingular projective variety X,*

$$\chi(X, \mathscr{E}) = \int_X \mathrm{ch}(\mathscr{E}) \cdot \mathrm{td}(\mathcal{T}_X).$$

Metaproof. See [Har, Theorem A.4.1] for just the statement and further references. There is a sketch of proof in [Gat, Theorem 10.4.5]. For a full proof, see [Ful1, Corollary 15.2.1]. □

Chapter 2

Constantly Branched Coverings

In this chapter, we study a class of coverings $\pi : Y \to X$ of varieties, which we will call *constantly branched* along an arrangement H of hypersurfaces in X. In [BHH], Hirzebruch had introduced this notion for complex surfaces, and we now generalize it significantly. In the nonsingular case, we derive formulas relating two important numerical invariants of these varieties, namely the Euler characteristic (over $\Bbbk = \mathbb{C}$) and the self-intersection number of a canonical divisor. These relations will depend mainly on combinatorial data of H.

In Section 2.5, we prove that any such covering can be desingularized by a simple sequence of blow-ups and in Section 2.6, we construct constantly branched coverings associated to a certain class of arrangements. In particular, arrangement of hyperplanes in projective space will belong to this class. In the case of surfaces, these results and constructions specialize to what Hirzebruch already described in [BHH].

2.1 Ramified and Unramified Morphisms

In this section, we recall several definitions and results from the study of morphisms $\pi : Y \to X$ of finite type. This family of morphisms is the algebraic equivalent of branched coverings. If Y and X are varieties, π corresponds to an algebraic extension of fields $\Bbbk(X) \hookrightarrow \Bbbk(Y)$. The degree of this extension is also called the **degree of** π, denoted by $\deg(\pi)$. It is equal to the cardinality of the generic fibers of π. The closed set where the fibers are of smaller cardinality is the ramification locus of the covering. We now make this notion formal.

Definition 2.1. *Let X and Y be Noetherian schemes and let $\pi : Y \to X$ be a morphism of finite type. Let $Q \in Y$ be any point and set $P := \pi(Q)$. We say that π is* ***unramified at*** *Q if $\pi^\sharp_Q : \mathcal{O}_{X,P} \to \mathcal{O}_{Y,Q}$ satisfies $\mathfrak{m}_P \cdot \mathcal{O}_{Y,Q} = \mathfrak{m}_Q$. Otherwise, we say that π is* ***ramified*** *at Q. We denote by $\mathcal{R}_\pi \subseteq Y$ the set of points where π is ramified and call it the* ***ramification locus of*** *π. The set $\mathcal{B}_\pi := \pi(\mathcal{R}_\pi)$ is called the* ***branch locus*** *of π. The morphism π is called* ***unramified*** *if it is nowhere ramified.*

Example 2.1.1. *A good example for intuition is the projection of a parabola to the ordinate, as sketched in Figure 2.1.*

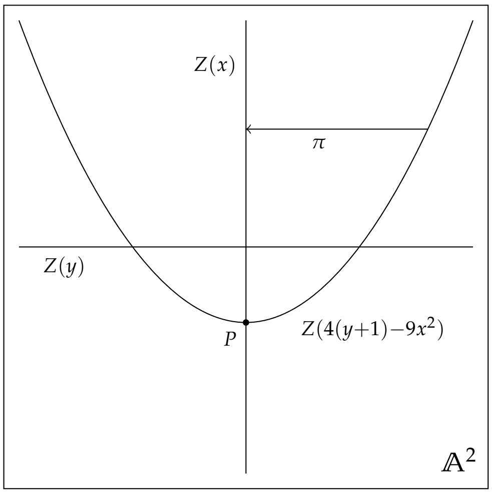

Figure 2.1: Projecting from a parabola

It is an example for a morphism of degree two. P is the only ramification (branching) point on the parabola (ordinate).

We quote the famous result of Oscar Zariski from 1958 which assures that $\mathcal{B}_\pi$ and $\mathcal{R}_\pi$ can be understood as effective divisors:

Theorem 2.2 (Purity of the Branch Locus). *If $\pi : Y \to X$ is a morphism of finite type between varieties, $\mathcal{R}_\pi$ and $\mathcal{B}_\pi$ are pure*[1] *of codimension one.*

Metaproof. By [Zar, Proposition 2], the set $\mathcal{R}_\pi$ is closed and pure of codimension one. Since a finite morphism maps points of codimension one to points of codimension one, the same holds for $\mathcal{B}_\pi$. □

Definition 2.3. *Let $\pi : Y \to X$ be a morphism of finite type between varieties over a field $\Bbbk$. Let $Q \in Y$ be a closed point and set $P := \pi(Q)$. Let*

$$Y_P := Y \times_X \operatorname{Spec}(\Bbbk(P))$$

[1]Being **pure** means that all irreducible components have the same dimension.

*be the **scheme-theoretic fiber** of P under π. It is well-known that $\mathrm{sp}(Y_P)$ is homeomorphic to $\pi^{-1}(P)$, see [Har, Exercise II.3.10]. We define*

$$e_\pi(Q) := \mathrm{len}\left(\mathcal{O}_{Y_P,Q}\right)$$

*and call it the **ramification index of** π **at** Q. If $Z = \overline{Q}$ is the closure of Q, we also write $e_\pi(Z) := e_\pi(Q)$. If $\mathrm{char}(\Bbbk)$ divides $e_\pi(Q)$, we say that the ramification is **wild**, otherwise it is **tame**.*

The following Proposition 2.4 explains the connection between ramification index and the notion of π being ramified:

Proposition 2.4. *With notation as in Definition 2.3, $\mathcal{O}_{Y,Q}/(\mathfrak{m}_P \cdot \mathcal{O}_{Y,Q}) \cong \mathcal{O}_{Y_P,Q}$.*

Proof. We may assume that $X = \mathrm{Spec}(A)$ and hence, $Y = \pi^{-1}(X) = \mathrm{Spec}(B)$ is also affine. By definition, $\mathcal{O}_{Y_P,Q} = (B \otimes_A \Bbbk(P))_Q$. Furthermore,

$$\begin{array}{rcl} (B \otimes_A \Bbbk(P))_Q = (B \otimes_A (A_P/\mathfrak{m}_P))_Q & \xrightarrow{\sim} & B_Q/(\mathfrak{m}_P \cdot B_Q) \\ \dfrac{b \otimes (a \bmod \mathfrak{m}_P)}{h} & \longmapsto & \dfrac{a \cdot b}{h} \bmod (\mathfrak{m}_P \cdot B_Q) \end{array}$$

is an isomorphism: For injectivity, $ab \in \mathfrak{m}_P B_Q$ implies $ab = a'b'$ with $a' \in \mathfrak{m}_P$ and $b' \in B_Q$, but then $b \otimes (a \bmod \mathfrak{m}_P) = b' \otimes (a' \bmod \mathfrak{m}_P) = 0$. □

Corollary 2.5. *Let $\pi : Y \to X$ be a morphism of finite type between integral schemes. Let $Q \in Y$ and $P := \pi(Q)$. Then, $e_\pi(Q) = 1$ if and only if π is unramified at Q.*

Proof. Note that $e_\pi(Q) = 1$ if and only if $\mathcal{O}_{Y_P,Q}$ is a field, i.e. if and only if it is equal to $\Bbbk(Q)$. By Proposition 2.4, this is equivalent to

$$\mathcal{O}_{Y,Q}/(\mathfrak{m}_P \cdot \mathcal{O}_{Y,Q}) = \mathcal{O}_{Y_P,Q} = \Bbbk(Q) = \mathcal{O}_{Y,Q}/\mathfrak{m}_Q\,.$$ □

The following corollary connects our definition of the ramification index with the one given in [Har, IV.2]:

Corollary 2.6. *Let $\pi : Y \to X$ be a finite, dominant morphism of regular integral schemes. Let $Q \in Y$ be a point of codimension one and $P := \pi(Q)$. Let f be a uniformizing parameter at P, i.e. $\mathfrak{m}_P = (f)$. Let $v_Q : \Bbbk(Y) \to \mathbb{Z}$ denote the valuation corresponding to $\mathcal{O}_{Y,Q}$. Then, $e_\pi(Q) = v_Q(\pi_Q^\sharp(f))$.*

Proof. By [Eis, Proposition 11.1], since Y is regular, v_Q can be evaluated on $\mathcal{O}_{Y,Q}$ as follows: If g is a uniformizing parameter at Q, i.e. $\mathfrak{m}_Q = (g)$, then any element $\alpha \in \mathcal{O}_{Y,Q}$ can be written as $\alpha = ug^v$ for some unit u and $v = v_Q(\alpha)$. Let $e := v_Q(\pi_Q^\sharp(f))$, then Proposition 2.4 yields

$$\mathcal{O}_{Y_P,Q} = \mathcal{O}_{Y,Q}\Big/\left(\pi_Q^\sharp(f)\right) = \mathcal{O}_{Y,Q}/(g^e)\,,$$

which is easily seen to have length e over itself. □

Remark 2.6.1. *By Theorem 2.2 and Corollary 2.5, there is a finite number of points Q of codimension one where π is ramified, and these are the points with $e_\pi(Q) > 1$.*

Definition 2.7. *Let $\pi : Y \to X$ be a morphism of finite type between $\Bbbk$-varieties and $Q \in Y$ a closed point. Let $P := \pi(Q)$, then we call*

$$f_\pi(Q) := [\Bbbk(Q) : \Bbbk(P)]$$

*the **inertia degree** of π at Q. This is the degree of the restricted morphism $\overline{Q} \to \overline{P}$.*

A very important tool in the analysis of branched coverings will be the following formula:

Theorem 2.8 (Degree Formula). *Let $\pi : Y \to X$ be a finite, dominant morphism of integral regular schemes. Then, for any closed point $P \in X$,*

$$\deg(\pi) = \sum_{\pi(Q)=P} e_\pi(Q) \cdot f_\pi(Q)$$

Metaproof. This is [GW, Formula (12.6.2), Page 329]. Note that π is flat because X and Y are regular, see [Liu, Remark 4.3.11]. □

As one application, we can show that an unramified morphism has constant fiber cardinality:

Corollary 2.9. *Let $\pi : Y \to X$ be an unramified, finite and surjective morphism of nonsingular $\Bbbk$-varieties over an algebraically closed field $\Bbbk$. Then,*

$$\left|\pi^{-1}(P)\right| = \deg(\pi)$$

for each closed point $P \in X$.

Proof. Since π is unramified, Corollary 2.5 and Theorem 2.8 imply

$$\deg(\pi) = \sum_{\pi(Q)=P} [\Bbbk(Q) : \Bbbk(P)] = \sum_{\pi(Q)=P} 1 = \left|\pi^{-1}(P)\right|.$$

Note that $\Bbbk(Q) \cong \Bbbk(P) \cong \Bbbk$ since $\Bbbk$ is algebraically closed. □

2.2 Constantly Branched Coverings

We will now restrict to a special class of finite morphisms. These *constantly branched coverings* will be the objects of our study for the rest of the chapter. Their branch locus is required to be a so-called *strict arrangement*. For our later applications, it might serve intuition well to picture arrangements of hyperplanes in general position, which are always strict in the following sense:

Definition 2.10. *An effective divisor H inside a nonsingular variety X will be called an* ***arrangement*** *if $H = H_0 + \ldots + H_\ell$ such that the H_i are prime and for any $\lambda \subset \{0, \ldots, \ell\}$, the scheme-theoretic intersection*

$$H_\lambda := \prod_{i \in \lambda} H_i$$

is a nonsingular subvariety of X. We say that H is a ***strict arrangement*** *if, in addition, the H_i intersect transversally – or, equivalently, H has normal crossings. See also [Liu, Definition 9.1.6]. For any point $P \in X$, closed or not, we define*

$$\lambda_H(P) := \{ i \in \{0, \ldots, \ell\} \mid P \in H_i \} \quad \text{and} \quad r_H(P) := |\lambda_H(P)| .$$

*We write $\lambda(P)$ and $r(P)$ if there is no ambiguity concerning H. We also say that P is an r-**point of** H when we mean $r := r(P)$.*

Example 2.10.1. *As mentioned in the introduction, our main example is the case where the $H_i = Z_*(h_i) \subset \mathbb{P}^s$ are hyperplanes. In other words, the h_i are linear homogeneous polynomials in $s+1$ variables. The scheme-theoretic intersection H_λ corresponds to the ideal $(h_i \mid i \in \lambda)$, which is radical. Hence, H_λ is a linear subvariety and as such, also nonsingular. If $s = 2$, H is a set of projective lines and an r-point of H is a (closed) point in the projective plane where r lines intersect.*

The arrangements we are interested in will be the geometric duals of SGCs. Inevitably, we will have more than d points lie inside a linear subvariety of dimension $d-1$. In the dual setting, we will have more than d hyperplanes intersect in a variety of codimension d. These parts of the arrangement will be exactly the parts that we have to blow up in Section 2.5 to regularize the covering.

Definition 2.11. *Let H be an arrangement inside a nonsingular $\mathbb{k}$-variety X. An intersection $H_\lambda \neq \emptyset$ is* ***redundant*** *if $\operatorname{codim}_X(H_\lambda) < |\lambda|$. In this case, we also call λ redundant. A point $P \in X$ will be called H-**redundant** if $\lambda_H(P)$ is redundant. Note that by definition, the set of H-redundant points is a closed subvariety of codimension two, which we denote by $\operatorname{Rd}(H)$ and refer to as the* ***redundant part of*** *H.*

Example 2.11.1. *In the situation of Example 2.10.1 with $s = 2$, the redundant intersections are those points in the plane where more than two lines meet.*

Fact 2.12. *Let H be an arrangement inside a nonsingular variety X. An intersection H_λ is redundant if and only if there exists an $i \in \lambda$ such that $H_\lambda = H_{\lambda \setminus \{i\}}$. There are $d := \operatorname{codim}_X(H_\lambda)$ components of H which intersect transversally at the generic point P of H_λ.*

Proof. The implication "$\Leftarrow$" is obvious, so assume that H_λ is redundant. For ease of notation, let us assume that $\lambda = \{1, \ldots, r\}$ and $X = \operatorname{Spec}(A)$ is affine. By localizing further, we may assume that the $I(H_i) = (h_i)$ are principal ideals. By definition of H_λ as a scheme-theoretic intersection, $\mathfrak{m}_P = (h_1, \ldots, h_r)$. Let $\bar{h}_i$ be the image of h_i under $\mathfrak{m}_P \twoheadrightarrow \mathfrak{m}_P/\mathfrak{m}_P^2$. Since X is nonsingular, we may assume that $\{\bar{h}_1, \ldots, \bar{h}_d\}$ is a $\mathbb{k}(P)$-basis for $\mathfrak{m}_P/\mathfrak{m}_P^2$. By Nakayama's Lemma [Eis, Corollary 4.8], this implies $\mathfrak{m}_P = (h_1, \ldots, h_d)$. $\square$

Remark 2.12.1. *Note that the redundant part is, in general, not pure of codimension two – it might happen that no three of the $\bar{h}_i$ are linearly dependent, but any four of them are.*

Corollary 2.13. *Let H be an arrangement inside a nonsingular variety X. Then H is strict if and only if it has no redundant intersections. In this case, $r(P) \leq \operatorname{codim}_X(P)$ for each $P \in X$.* $\square$

Remark 2.13.1. *Note that in the situation of Example 2.10.1, the transversality condition is obvious: Any two distinct hyperplanes intersect transversally.*

Notation 2.14. *We denote by $t_r(d, H)$ the number of r-points P of codimension d such that $H_{\lambda(P)} = \overline{P}$, i.e. P is the generic point of the intersection of all components it is contained in. Note that this notation is in agreement with Definition 1.8 for the case where H is an arrangement of hyperplanes.*

We can now define what a constantly branched covering is. Recall that for a field K containing all n-th roots of unity, a Kummer extension is an algebraic extension of the form

$$K\,[\sqrt[n]{x_1}, \ldots, \sqrt[n]{x_\ell}\,]\,.$$

Our reference is [Bos, 4.9]. One usually assumes that $\operatorname{char}(K)$ does not divide n. In this case, the extension is automatically Galois.

Notation 2.15. *Let A be a domain and $K := \operatorname{Frac}(A)$. For any nonzero $x \in A$, we understand $\sqrt[n]{x}$ as a set. More precisely, $\sqrt[n]{x} = \{\, y \in \overline{K} \mid y^n = x \,\}$.*

Definition 2.16. *A finite surjective morphism $\pi : Y \to X$ of $\mathbb{k}$-varieties will be called a **covering** if $\mathcal{B}_\pi$ is an arrangement (this terminology is* not *standard). A covering is called **regular** if its branch locus is a strict arrangement.*

*A covering is called **n-fold locally Kummer** if $\operatorname{char}(\mathbb{k})$ does not divide $n \in \mathbb{N}$ and for any closed point $Q \in Y$, $P := \pi(Q)$, there exist $x_1, \ldots, x_\ell \in \mathcal{O}_{X,P}$ such that*

$$\mathcal{O}_{Y,Q} = \mathcal{O}_{X,P}\,[y_1, \ldots, y_\ell]$$

for certain $y_i \in \sqrt[n]{x_i}$. Checking this property at generic points, we see that $\Bbbk(Y)$ is a Kummer extension of $\Bbbk(X)$, therefore locally Kummer coverings are Galois.

*An **(n-fold) constantly branched** covering is defined to be an n-fold locally Kummer covering over a smooth base X such that in the above scenario, we also assume that $x_1, \ldots, x_r$ define the components of $\mathcal{B}_\pi$ near P and $x_i \in \mathfrak{m}_P$ if and only if $i \leq r$. In other words, $\mathcal{I}(\mathcal{B}_\pi)_P = (x_1 \cdots x_r)$.*

Notation 2.16.1. *We will write CBC instead of "constantly branched covering". Whenever the term is used, we will also implicitly assume that the base field $\Bbbk$ is algebraically closed.*

Remark 2.16.2. *If π is a CBC then in particular, each component of the branch locus has ramification index n. To see this, just choose a closed 1-point of $\mathcal{B}_\pi$. By assumption, the ramification is always tame.*

One important property of CBCs is the fact that we understand the singularities of Y very well:

Proposition 2.17. *If $\pi : Y \to X$ is a CBC with branch locus H, the closed singular points of Y are the closed points of $\pi^{-1}(\mathrm{Rd}(H))$. Hence,*

$$\mathrm{Sing}(Y) = \pi^{-1}(\mathrm{Rd}(H)).$$

Proof. Let P be a closed r-point and $Q \in \pi^{-1}(P)$. Let $\xi_1, \ldots, \xi_\ell \in \mathcal{O}_{X,P}$ such that $\mathcal{O}_{Y,Q} = \mathcal{O}_{X,P}[\psi_1, \ldots, \psi_\ell]$ with $\psi_i \in \sqrt[n]{\xi_i}$. Let $U = \mathrm{Spec}(A)$ be an affine neighborhood of P and $V := \pi^{-1}(U) = \mathrm{Spec}(B)$. Since B is a finitely generated A-algebra, we can assume $B = A[\psi_1, \ldots, \psi_\ell]$ by possibly localizing further. Also, we may assume that $\xi_i \in A^\times$ if and only if $i > r$.

Since X is a $\Bbbk$-variety, $A = \Bbbk[x_1, \ldots, x_d]/I$ is a finitely generated $\Bbbk$-algebra, and we pick generators $I = (g_1, \ldots, g_t)$ of the ideal I. We denote by $h_i \in \Bbbk[x_1, \ldots, x_d]$ a representative of $\xi_i \in A$. Let $f_i := h_i - y_i^n$, then

$$B = A[\psi_1, \ldots, \psi_\ell] = \Bbbk[x_1, \ldots, x_d, y_1, \ldots, y_\ell]/(g_1, \ldots, g_t, f_1, \ldots, f_\ell).$$

Note that $\partial_{y_i} f_j = -\delta_{ij} n y_i^{n-1}$ and $\partial_{y_i} g_j = 0$. By the Jacobian criterion, Y is nonsingular in Q if and only if the matrix

$$J_Y := \begin{pmatrix} \partial_{x_1} g_1 & \cdots & \partial_{x_d} g_1 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ \partial_{x_1} g_t & \cdots & \partial_{x_d} g_t & 0 & \cdots & 0 \\ \partial_{x_1} h_1 & \cdots & \partial_{x_d} h_1 & -n y_1^{n-1} & & \mathbf{0} \\ \vdots & \ddots & \vdots & & \ddots & \\ \partial_{x_1} h_\ell & \cdots & \partial_{x_d} h_\ell & \mathbf{0} & & -n y_\ell^{n-1} \end{pmatrix}$$

has rank $\ell + d - s$ at Q, where $s := \dim(Y)$. Note that

$$y_i(Q) = 0 \iff 0 = y_i^n(Q) = \xi_i(Q) = \xi_i(P) \iff i \leq r.$$

We set $b_i := -n \cdot y_i^{n-1}(Q)$ and note that b_i is nonzero if and only if $i > r$ since $\mathrm{char}(\Bbbk)$ does not divide n. Thus,

$$J_Y(Q) = \begin{pmatrix} (\partial_{x_1} g_1)(Q) & \cdots & (\partial_{x_d} g_1)(Q) & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ (\partial_{x_1} g_t)(Q) & \cdots & (\partial_{x_d} g_t)(Q) & 0 & \cdots & 0 & 0 & \cdots & 0 \\ (\partial_{x_1} h_1)(Q) & \cdots & (\partial_{x_d} h_1)(Q) & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ (\partial_{x_1} h_r)(Q) & \cdots & (\partial_{x_d} h_r)(Q) & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & b_{r+1} & & \mathbf{0} \\ \vdots & & \vdots & \vdots & & \vdots & & \ddots & \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \mathbf{0} & & b_\ell \end{pmatrix}.$$

Note that the upper left $(t + r) \times d$ – submatrix of $J_Y(Q)$ is the Jacobian J_Z of $Z := Z(\xi_1, \ldots, \xi_r) \subseteq X$, evaluated at P. In other words, Z is the intersection of the components of H passing through P. Since that intersection is nonsingular,

$$\begin{aligned} P \notin \mathrm{Rd}(H) &\Leftrightarrow \dim(Z) = s - r \\ &\Leftrightarrow \mathrm{rank}(J_Z(P)) = d - (s - r) = r + d - s \\ &\Leftrightarrow \mathrm{rank}(J_Y(Q)) = \ell + d - s. \\ &\Leftrightarrow Q \notin \mathrm{Sing}(Y) \end{aligned}$$ ☐

Corollary 2.18. *If $\pi : Y \to X$ is a regular CBC, then Y is nonsingular.* ☐

For better intuition, we give a basic example of a CBC over the affine plane, resulting from the adjunction of roots of linear forms. Ultimately, this is exactly the setting that we want to study.

Example 2.19. *Let $A := \Bbbk[x, y]$ and $A' := A[z_1, z_2, z_3]$. Set*

$$h_1 := x \qquad h_2 := y \qquad h_3 := x + 2$$

and define $B := A' / (z_i^n - h_i)$. We set $X := \mathbb{A}^2 = \mathrm{Spec}(A)$ and $Y := \mathrm{Spec}(B)$. Then, the integral extension $A \to B$ induces a finite morphism $\pi : Y \to X$. Let $P := (x, y) \subset A$ be the origin of $\mathbb{A}^2$. We note that the points $Q \in Y$ with $\pi(Q) = P$ are exactly the maximal ideals $Q_\alpha = (z_1, z_2, z_3 - \alpha)$ where $\alpha \in \sqrt[n]{2}$. In fact,

$$\bigcap_{\zeta^n = 2} Q_\zeta = (z_1, z_2) =: Q = \sqrt{PB} = I(\pi^{-1}(P)).$$

In some neighborhood of Q_α, the elements $z_3 - \zeta$ for any other $\alpha \neq \zeta \in \sqrt[n]{2}$ become units and since

$$\prod_{\zeta^n=2} (z_3 - \zeta) = h_3 - 2 = h_1 = z_1^n \qquad \Longrightarrow \qquad z_3 - \alpha = \frac{z_1^{n-1}}{\prod_{\zeta \neq \alpha}(z_3 - \zeta)} \cdot z_1,$$

this means that $\mathfrak{m}_{Q_\alpha} = (z_1, z_2)$ is indeed generated by two elements which are n-th roots of x and y, respectively. Similarly, we observe that over $P' := (x+2, y)$, the points are locally generated by z_2 and z_3.

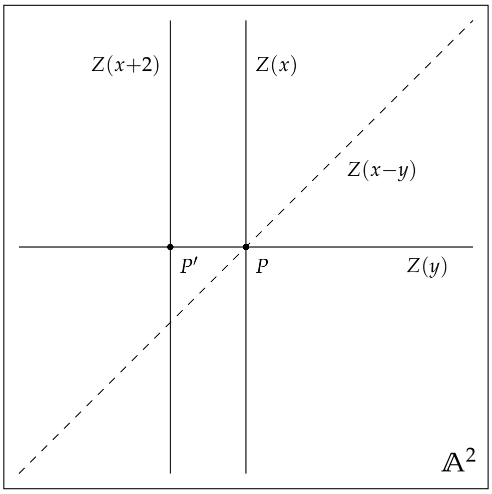

Figure 2.2: The branching locus of π in Example 2.19.

If we add the equation $h_4 := x - y$, as well as a z_4 with $z_4^n = h_4$, the h_i do not longer define a strict arrangement: P is a redundant intersection. In fact, we then have $Q = (z_1, z_2, z_4)$ and we will show later and in more generality that Q is not generated by any two of them (see Lemma 2.45 and Corollary 2.47). Similarly, all points Q_α are now singular, because $\mathfrak{m}_{Q_\alpha}$ can not be generated by two elements.

For the time being, we will only study the case of regular CBCs:

Fact 2.20. *Let $\pi : Y \to X$ be an n-fold regular CBC. For any closed r-point P of $\mathcal{B}_\pi$ and any $Q \in \pi^{-1}(P)$, there exists a local coordinate systems $x_1, \ldots, x_d \in \mathcal{O}_{X,P}$ and $y_1, \ldots, y_d \in \mathcal{O}_{Y,Q}$ such that*

(a). $\mathcal{I}(\mathcal{B}_\pi)_P = (x_1 \cdots x_r)$ and $\mathcal{I}(\mathcal{R}_\pi)_Q = (y_1 \cdots y_r)$.

(b). $x_i = y_i^n$ for $1 \leq i \leq r$ and $x_i = y_i$ otherwise.

We will write RCBC instead of "regular CBC".

Proof. We can find a coordinate system with the desired properties around P as Corollary 2.13 guarantees $\mathcal{B}_\pi$ to cross normally. Let $\xi_1, \ldots, \xi_\ell \in \mathcal{O}_{X,P}$ be such that

$$\mathcal{O}_{Y,Q} = \mathcal{O}_{X,P}\left[\psi_1, \ldots, \psi_\ell\right]$$

with $\psi_i^n = \xi_i$. We may assume that $\xi_i = x_i$ for $1 \leq i \leq r$. For $i > r$, we know that ξ_i is a unit. Consequently ψ_i is also invertible for $i > r$. Replacing $\mathcal{O}_{X,P}$ by $\mathcal{O}_{X,P}\left[\psi_{r+1}, \ldots, \psi_\ell\right]$, we may therefore assume that

$$\mathcal{O}_{Y,Q} = \mathcal{O}_{X,P}\left[y_1, \ldots, y_r\right]$$

where $y_i^n = x_i$. Consequently,

$$\mathfrak{m}_Q = \mathfrak{m}_P \cdot \mathcal{O}_{Y,Q} + (y_1, \ldots, y_r) = (y_1, \ldots, y_r, x_{r+1}, \ldots, x_d)\,. \qquad \square$$

2.3 Analytification and Euler Characteristic

When we talk about the Euler characteristic of a variety X, it would be fatal to think of the Euler characteristic of the topological space $\mathrm{sp}(X)$ that underlies the scheme structure: By [Ram, Theorem 4.14], the singular cohomology groups $H^q(X, \mathbb{Q})$ with coefficients in $\mathbb{Q}$ agree with the sheaf cohomology groups $\mathcal{H}^q(X, \mathbb{Q}_X)$, where $\mathbb{Q}_X$ denotes the constant sheaf $U \mapsto \mathbb{Q}$ on X. Since $\mathbb{Q}_X$ is flasque, [Har, Proposition III.2.5] yields

$$H^q(X, \mathbb{Q}) = \begin{cases} \mathbf{0} & ; \quad q > 0 \\ \mathbb{Q} & ; \quad q = 0 \end{cases}$$

By the universal coefficient theorems in homology and cohomology given in [Hat, Theorems 3A.3 and 3.2], we conclude

$$\chi(\mathrm{sp}(X)) = \mathrm{rank}(H_0(X, \mathbb{Z})) = \dim(H^0(X, \mathbb{Q})) = 1.$$

Hence, in this section, we assume $\Bbbk = \mathbb{C}$ and consider the Euler characteristic of the associated complex manifold: Our reference is mainly the very comprehensible [Wer], but for its basic properties one might also refer to [Har, Appendix B]. The **analytification functor** $(-)^{\mathrm{an}}$ associates to any complex, smooth, projective variety X the complex manifold X^{an} consisting of its closed points. We then simply write $\chi(X) := \chi(X^{\mathrm{an}})$.

Proposition 2.21. *If $\pi : Y \to X$ is an n-fold RCBC of degree N with branch locus $H := \mathcal{B}_\pi$, we define*

$$H(r) := X \setminus \bigcup\nolimits_{|\lambda| \neq r} H_\lambda = \{\, P \in X \mid r_H(P) = r \,\}$$

Then, for any component Z of $H(r)$ and any component W of $\pi^{-1}(Z)$, the morphism $\pi|_W : W \to Z$ is unramified of degree N/n^r.

Proof. Let $W_1 \cup \cdots \cup W_r = \mathcal{R}_\pi$ be the irreducible components of its ramification locus. Then,

$$\pi_i := \pi|_{W_i} : W_i \longrightarrow \pi(W_i)$$

is an n-fold RCBC with $\mathcal{R}_{\pi_i} = \bigcup_{j \neq i}(W_j \cap W_i)$ by the local description in Fact 2.20. By induction on r, this yields our claim. □

Proposition 2.22. *If $\pi : Y \to X$ is an unramified surjective morphism of degree N between smooth complex varieties, then π^{an} is an N-fold covering map. In particular, $\chi(Y) = N \cdot \chi(X)$.*

Proof. This follows from [Wer, Corollary 6.11] and Corollary 2.9. □

Proposition 2.23. *Let X be a complex, smooth variety and $Y \subseteq X$ a closed subvariety. Let $U := X \setminus Y$, then $\chi(X) = \chi(Y) + \chi(U)$.*

Metaproof. Solve the exercise on page 95 in [Ful2]. Alternatively, look up the solution on page 141. Intuitively, the reason for this result is that Y is a neighborhood retract of X in the classical topology – application of Mayer-Vietoris then yields the desired result. □

We obtain the following important result, which will be our main tool for calculating the Euler characteristic of CBCs:

Corollary 2.24. *Let $\pi : Y \to X$ be an n-fold RCBC of complex algebraic varieties with branch locus H. Let $N := \deg(\pi)$, then*

$$\chi(Y) = \sum_{r \in \mathbb{N}} \frac{N \cdot \chi(H(r))}{n^r}$$

Proof. This follows directly from Propositions 2.21 to 2.23. □

2.4 Canonical Divisors

The canonical divisor of a complex variety is the determinant bundle of holomorphic n-forms. More generally, it is the dualizing object for Serre duality and consequently, an important object of study. Its inverse can also be understood as the first Chern class (c.f. Proposition 3.17), so it is of particular interest for us.

We study the behavior of canonical divisors under constantly branched coverings. Let us recall some definitions from [Har, II.8]:

Definition 2.25. *Let X be a smooth variety of dimension n and let $\delta : X \to X \times X$ be the diagonal[2] morphism. Let $\Delta := \delta(X)$ be the diagonal and $\mathcal{I}$ the ideal sheaf of Δ in $X \times X$. Then, the **sheaf of relative differentials of** X is defined to be*

$$\Omega_X := \delta^*(\mathcal{I} \big/ \mathcal{I}^2).$$

Its dual

$$\mathcal{T}_X := \Omega_X^\vee = \mathscr{H}om_{\mathcal{O}_X}(\Omega_X, \mathcal{O}_X)$$

*is called the **tangent sheaf of** X and the **canonical sheaf of** X is defined to be its maximal exterior power*

$$\omega_X := \bigwedge^n \Omega_X.$$

*Note that ω_X is an invertible sheaf on X. A **canonical divisor on** X is any Cartier divisor K_X which corresponds to ω_X.*

For a CBC $\pi : Y \to X$, we are going to express K_Y in terms of the pull-backs of K_X and the branching locus $\mathcal{B}_\pi$. Later on, K_X will be a well-known quantity since we work over $X = \mathbb{P}^s$ and likewise, we will have a good combinatorial understanding of the arrangement $\mathcal{B}_\pi$, which will consist only of hyperplanes.

Theorem 2.26 (Ramification Formula). *Let $\pi : Y \to X$ be a dominant morphism of finite type between nonsingular varieties that ramifies tamely. Denote by K_X and K_Y canonical divisors on X and Y, respectively. Then,*

$$K_Y \sim \pi^*(K_X) + \sum_{\operatorname{codim}_Y(Z)=1} (e_\pi(Z) - 1) \cdot Z.$$

Metaproof. Although the treatment in [Har] is for curves only, every statement up to [Har, Proposition IV.2.3] in that section is applicable to the case of nonsingular varieties and points of codimension one. Also recall that Corollary 2.6 identifies the ramification index in the reference with the one from Definition 2.3. □

Corollary 2.27. *If $\pi : Y \to X$ is an n-fold RCBC,*

$$K_Y \sim \pi^*(K_X) + \tfrac{n-1}{n} \cdot \pi^*(\mathcal{B}_\pi).$$

Proof. Since $e_\pi \equiv n$ on components of $\mathcal{R}_\pi$ and otherwise $e_\pi \equiv 1$, Theorem 2.26 yields

$$K_Y \sim \pi^*(K_X) + \sum_{\operatorname{codim}_Y(Z)=1} (e_\pi(Z) - 1) \cdot Z = \pi^*(K_X) + (n-1) \cdot \mathcal{R}_\pi.$$

Also, $\pi^*(\mathcal{B}_\pi) = n \cdot \mathcal{R}_\pi$ by the local description in Fact 2.20. □

[2] The diagonal morphism is $\delta := \mathrm{id}_X \times \mathrm{id}_X$, so it satisfies $\delta(x) = (x, x)$ on closed points.

Proposition 2.28. *Let $\pi : Y \to X$ be a finite surjective morphism of nonsingular varieties. Then, the composite*

$$A(X) \xrightarrow{\pi^*} A(Y) \xrightarrow{\pi_*} A(X)$$

is multiplication by $N := \deg(\pi)$. In particular, for all $\alpha \in A^0(X)$,

$$\int_Y \pi^*(\alpha) = \deg(\pi) \cdot \int_X \alpha.$$

Proof. The first statement is [Ful1, Example 1.7.4] and also follows from Theorem 2.8. Note that for any point $P \in X \setminus \mathcal{B}_\pi$, we have $\left|\pi^{-1}(P)\right| = N$ by Corollary 2.9. Therefore, $\int_Y \pi^*[P] = N$. Hence, for any $\sum_i n_i P_i \in Z^{\dim(X)}(X)$ which maps to α, we have to "move" the points P_i out of the branch locus of π. More precisely, we have to show that for any $P \in \mathcal{B}_\pi$, the cycle $[P]$ is rationally equivalent to some $[P']$ with $P' \notin \mathcal{B}_\pi$.

To do so, we can just choose a general (nonsingular) curve $C \subset X$ which is not a component of $\mathcal{B}_\pi$ and which passes through P. Let $P' \in C \setminus \mathcal{B}_\pi$. We choose uniformizing variables $f \in \mathcal{O}_{C,P}$ and $f' \in \mathcal{O}_{C,P'}$. Then, the function

$$\phi := f/f' \in \Bbbk(C)$$

satisfies $\operatorname{div}(\phi) = [P] - [P']$ as desired. $\square$

Putting it all together now yields a formula for the self-intersection number of a canonical divisor on Y.

Corollary 2.29. *Let $\pi : Y \to X$ be an n-fold RCBC and $s := \dim(Y)$. Then,*

$$\int_Y [K_Y]^s = \deg(\pi) \cdot \int_X \left([K_X] + \tfrac{n-1}{n} \cdot [\mathcal{B}_\pi]\right)^s$$

Proof. Follows from Corollary 2.27 and Proposition 2.28. $\square$

2.5 Singular Case and Regularization

In our effort to prove Sylvester-Gallai bounds, we will construct constantly branched coverings $\pi : Y \to \mathbb{P}^s$, branched along an arrangement H of hyperplanes which is dual to an SG_k-closed set of points. By definition of such a set, H will always have redundant intersections. Hence, the covering will not be regular. The best result we can hope for is a way to transform such a covering into a regular one, resolving the singularities of Y. We prove that this is always possible by blowing up redundant intersections and their preimages. This is a generalization of the methods described in [BHH, Chapter 1.2] to arbitrary dimension and (algebraically closed) base field.

The key observation is that we do not have to blow up in $\pi^\star(\mathcal{I}(H_\lambda))$, but may actually blow up in the ideal sheaf of $\pi^{-1}(H_\lambda)$:

Lemma 2.30. *Let $X = \operatorname{Spec}(A)$, $Y = \operatorname{Spec}(B)$ and $\varphi : Y \to X$ a finite morphism. Assume that $I = (x_1, \ldots, x_r) \subseteq A$ and $J = (y_1, \ldots, y_r) \subseteq B$ are ideals satisfying $y_i^n = x_i$ for some $n \in \mathbb{N}$ and all i. Then,*

$$\begin{tikzcd} \mathrm{Bl}_J(Y) \arrow[r, dotted, "\exists!\bar{\varphi}"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Bl}_I(X) \arrow[d, "\beta"] \\ Y \arrow[r, "\varphi"'] & X \end{tikzcd}$$

Proof. By Corollary 1.28, we have to verify that under $\varphi \circ \alpha$ (corresponding to the inclusion $A \hookrightarrow B[JT]$), the ideal $I' := I \cdot B[JT]$ is invertible. In $(B[JT]_{y_iT})_0$, we can write

$$\frac{x_j T^n}{(y_i T)^n} \cdot x_i = \frac{x_j T^n}{x_i T^n} \cdot x_i = x_j,$$

so $(I'_{y_iT})_0 = (x_i)$ is principal for each i, proving that I' is locally principal. $\square$

The purpose of following two lemmata is to verify that $(y_1, \ldots, y_r)$ is, in fact, the ideal sheaf of $\pi^{-1}(H_\lambda)$.

Lemma 2.31. *Let A be a domain and let $I = (x_1, \ldots, x_\ell) \subseteq A$ be a radical ideal. Let $n \in \mathbb{N}$ and set $B := A[T_1, \ldots, T_\ell]/(T_i^n - x_i)$. We let $y_i \in B$ denote an image of T_i under the canonical projection. Then, $J := (y_1, \ldots, y_\ell) = \sqrt{IB}$.*

Proof. Clearly, $IB \subseteq J \subseteq \sqrt{IB}$. If J is radical, we are done. Let $f \in B$ be any element that satisfies $f^m \in J$ for some $m \in \mathbb{N}$. We can write it as a polynomial expression

$$f = \sum_{\nu = (\nu_1, \ldots, \nu_\ell)} \alpha_\nu \cdot y_1^{\nu_1} \cdots y_k^{\nu_\ell} \qquad \text{with} \qquad \alpha_\nu \in A.$$

Clearly, we only have to show $\alpha_{\mathbf{0}} = \alpha_{(0,\ldots,0)} \in J$. Because any term in f^m other than $\alpha_{\mathbf{0}}^m$ is of the form by_i for some $b \in B$, we know $\alpha_{\mathbf{0}}^m \in J \cap A = I$. Since I is a radical ideal of A, $\alpha_{\mathbf{0}} \in I = J \cap A$. $\square$

Fact 2.32. *Let $R = (R, \mathfrak{m})$ be a regular local ring. Then, it is equivalent for R to be of dimension zero, to be reduced and being a field.*

Proof. Regular local rings of dimension zero and fields are the same. If R is reduced, then $(0) = \sqrt{(0)} = \bigcap_{P \in \operatorname{Spec}(R)} P = \mathfrak{m}$, so R is a field. $\square$

Lemma 2.33. *Let A be a commutative ring, $I \subseteq A$ a radical (resp. maximal) ideal and $x \in A$ an element which is not contained in any prime that is minimal over I. Set $B := A[T]/(T^n - x)$ for some $n \in \mathbb{N} \cap A^\times$. Then, IB is radical (resp. maximal).*

Proof. Let $\pi : A[T] \twoheadrightarrow B$ be the canonical projection and $y := \pi(T)$. We want to show that

$$B/IB = (A/I)\,[T]/(T^n - x)$$

is reduced (resp. a field). Replacing A by A/I, we may assume $I = (0)$, A is reduced (resp. a field) and x is not contained in any minimal prime of A. In the case where I was maximal, it is obvious that $A[y]$, as an integral extension, is a field. Otherwise, we need to show that $B = A[y]$ is reduced. By [Liu, Exercise 2.8.2], this is equivalent to

(R_0) If Q is a minimal prime ideal of B, then the localization B_Q is a field. Here, we also use Fact 2.32.

(S_1) For any other prime ideal Q of B, $\operatorname{depth}(B_Q) > 0$.

Let $Q \in \operatorname{Spec}(B)$. For ($R_0$), assume that Q is minimal. Then, we know that $x \notin P := Q \cap A$. Thus, $x \in A_P^\times$ and consequently, $y \in B_Q^\times$. Since A_P is a field whose characteristic does not divide n, we can see that $B_Q = A_P[y]$ is also a field.

To verify property (S_1), we can assume $\dim(B_Q) > 0$. Since B is an integral extension of A, we have $\dim(A_P) = \dim(B_Q) > 0$. Since A is reduced, it satisfies (S_1), so there exists an element $a \in A_P$ which is not a unit and not a zero-divisor. Now, B is flat as an A-module because it is free. Thus, B_Q is flat over A_P and hence, a is not a zero-divisor in B_Q. This finishes the proof. $\square$

Corollary 2.34. *Let $\pi : Y \to X$ be an n-fold CBC with branch locus H and P the generic point of a component $\overline{P}$ of $\operatorname{Rd}(H)$. There exists an affine neighborhood $U = \operatorname{Spec}(A)$ of $P = (x_1, \ldots, x_r) \subset A$ such that, with $V := \pi^{-1}(U) = \operatorname{Spec}(B)$, we have $I(\pi^{-1}(\overline{P})) = (y_1, \ldots, y_r)$ for certain $y_i \in \sqrt[n]{x_i}$.* $\square$

Theorem 2.35 (Regularization). *Let $\pi : Y \to X$ be an n-fold CBC. Then, there exists a commutative diagram*

$$\begin{tikzcd}
\tilde{Y} \arrow[r, equal] \arrow[d, "\tilde{\pi}"] & Y_m \arrow[r, "\beta_m"] \arrow[d, "\pi_m"] & Y_{m-1} \arrow[r, "\beta_{m-1}"] \arrow[d, "\pi_{m-1}"] & \cdots \arrow[r, "\beta_2"] & Y_1 \arrow[r, "\beta_1"] \arrow[d, "\pi_1"] & Y_0 \arrow[r, equal] \arrow[d, "\pi_0"] & Y \arrow[d, "\pi"] \\
\tilde{X} \arrow[r, equal] & X_m \arrow[r, "\alpha_m"'] & X_{m-1} \arrow[r, "\alpha_{m-1}"'] & \cdots \arrow[r, "\alpha_2"'] & X_1 \arrow[r, "\alpha_1"'] & X_0 \arrow[r, equal] & X
\end{tikzcd} \tag{2.1}$$

such that the following properties hold:

(a). Each π_i is an n-fold CBC with branch locus $\mathcal{B}_{\pi_i} = \alpha_i^{-1}(\mathcal{B}_{\pi_{i-1}})$.

(b). Each α_{i+1} is the blow-up of X_i along a redundant intersection P_i of $\mathcal{B}_{\pi_i}$ and β_{i+1} is the blow-up along $\pi_i^{-1}(P_i)$.

We set $\beta := \beta_1 \circ \cdots \circ \beta_m$ and $\alpha := \alpha_1 \circ \cdots \circ \alpha_m$.

(c). The branching locus of $\tilde{\pi}$ is a strict arrangement.

(d). The morphism β is a resolution of singularities, i.e. $\tilde{Y}$ is a nonsingular variety and β is an isomorphism outside the singular locus of Y.

*Consequently, $\tilde{\pi}$ is an n-fold RCBC. We call $\tilde{\pi}$ a **regularization** of π.*

Proof. Let $H := \mathcal{B}_\pi$ and $P \in X$ be the generic point of a component of $\mathrm{Rd}(H)$. Since blowing up is local around P, we may assume that $X = \mathrm{Spec}(A)$ and $\mathrm{Spec}(B) = Y$ are affine. Furthermore, we can assume that $x_1, \ldots, x_r \in A$ define the components of H near P. The ideal $Q := \sqrt{PB}$ is the ideal of the preimage of P under π. Since $P = (x_1, \ldots, x_r)$, we can assume $Q = (y_1, \ldots, y_r)$ with $y_i^n = x_i$. Let $\tilde{Y}$ and $\tilde{X}$ be the blow-ups of Y and X along Q and P, respectively. By Lemma 2.30, we obtain a unique induced map $\tilde{\pi} : \tilde{Y} \to \tilde{X}$, corresponding to the following commutative diagram of graded $\Bbbk$-algebras:

$$\begin{tikzcd} B \arrow[r, hook, "\beta^\sharp"] & B[QT] \\ A \arrow[u, hook, "\pi^\sharp"] \arrow[r, hook, "\alpha^\sharp"'] \arrow[ur, phantom, "\circlearrowleft"] & A[PT] \arrow[u, hook, "\tilde{\pi}^\sharp"'] \end{tikzcd}$$

Here, $\tilde{X} = \mathrm{Proj}(A[PT])$ and $\tilde{Y} = \mathrm{Proj}(B[QT])$. By assumption, P defines a nonsingular, closed subvariety of X and therefore, $\tilde{X}$ is nonsingular by Theorem 1.33.

Since $\tilde{\pi}^\sharp$ is an integral extension of rings, $\tilde{\pi}$ is a finite morphism. We claim that its branch locus is $\tilde{H} := \alpha^{-1}(H)$. If we let $H_i := Z(h_i)$ denote the components of H, then $\tilde{H} = \tilde{H}_0 + \cdots + \tilde{H}_r$ where $\tilde{H}_i = \alpha^\intercal(H_i)$ is the strict transform of H_i for $i > 0$ and $\tilde{H}_0 = E_P$ is the exceptional divisor. We now show that any component of

$$E_Q = \tilde{\pi}^{-1}(E_P)$$

has ramification index n under $\tilde{\pi}$. Let $\tilde{Q}$ be the homogeneous ideal defining such a component. Since $\tilde{Q}$ is not irrelevant, there must be an index i such that $y_i T \notin \tilde{Q}$. Localizing in $y_i T$, we can conclude that

$$\frac{fT}{y_i T} \cdot y_i = f$$

for all $f \in Q = \tilde{Q}_0$. In other words, y_i is a uniformizer at $\tilde{Q}$. Since $x_i = y_i^n$ is a uniformizer at $\tilde{H}_0$, it follows that $e_{\tilde{\pi}}(\tilde{Q}) = n$.

To show that $\tilde{H}$ is an arrangement, pick any multiindex λ. By Corollary 1.32.(a), the intersection

$$\tilde{H}_\lambda = \alpha^\intercal(H_\lambda) = \mathrm{Bl}(H_\lambda, P)$$

is smooth because it is the blow-up of a nonsingular variety along another nonsingular, closed subvariety, see Theorem 1.33. Its intersection with $\tilde{H}_0$ is also smooth because it is the corresponding exceptional divisor.

To see that $\tilde{\pi}$ is a CBC, let $\tilde{Q} \in E_Q$ be a closed point, $\tilde{P} := \tilde{\pi}(\tilde{Q})$, $Q' := \beta(\tilde{Q})$ and $P' := \pi(Q') = \alpha(\tilde{P})$. Assume that $\tilde{P}$ is a t-point of $\tilde{H}$. We want to show that

$$\mathcal{O}_{\tilde{Y},\tilde{Q}} = \mathcal{O}_{\tilde{X},\tilde{P}}\,[\tilde{y}_1, \ldots, \tilde{y}_\ell]$$

for certain $\tilde{y}_i^n = \tilde{x}_i \in \mathcal{O}_{\tilde{X},\tilde{P}}$ and $\tilde{x}_i \in \mathfrak{m}_{\tilde{P}}$ if and only if it defines a component of $\tilde{H}$. Consider

$$\begin{tikzcd}
A \arrow[d, hook] & B \arrow[d, hook] & \\
\mathcal{O}_{X,P'} \arrow[r, hook] \arrow[d, hook] & \mathcal{O}_{Y,Q'} \arrow[r, equal] \arrow[d, hook] & \mathcal{O}_{X,P'}\,[\psi_1, \ldots, \psi_\ell] \arrow[dd, hook] \\
\mathcal{O}_{\tilde{X},\tilde{P}} \arrow[r, hook] \arrow[d, hook] & \mathcal{O}_{\tilde{Y},\tilde{Q}} \arrow[d, hook] & \\
\Bbbk(X) \arrow[r, hook] \arrow[d, equal] & \Bbbk(Y) \arrow[r, equal] \arrow[d, equal] & \Bbbk(X)[\psi_1, \ldots, \psi_\ell] \\
\mathrm{Frac}(A) & \mathrm{Frac}(B) &
\end{tikzcd}$$

where ψ_i are n-th roots of $\xi_1, \ldots, \xi_\ell \in \mathcal{O}_{X,P'}$. By Definition 2.16, we may assume that $\xi_i \in \mathfrak{m}_{P'} \Leftrightarrow \xi_i = x_i \Leftrightarrow i \leq r$. Clearly,

$$\mathcal{O}_{\tilde{X},\tilde{P}}\,[\psi_1, \ldots, \psi_\ell] \subseteq \mathcal{O}_{\tilde{Y},\tilde{Q}} \subseteq \mathrm{Frac}\left(\mathcal{O}_{\tilde{X},\tilde{P}}\,[\psi_1, \ldots, \psi_\ell]\right).$$

Replacing $\mathcal{O}_{\tilde{X},\tilde{P}}$ by $\mathcal{O}_{\tilde{X},\tilde{P}}\,[\psi_{r+1}, \ldots, \psi_\ell] = \mathcal{O}_{\tilde{X},\tilde{P}}(\psi_{r+1}, \ldots, \psi_\ell)$, we may henceforth assume that $\ell = r$, $\xi_i = x_i$ and $\psi_i = y_i$ for all i. Note that

$$\mathcal{O}_{\tilde{Y},\tilde{Q}} = \mathcal{O}_{Y,Q'}\left[\tfrac{a}{b} \;\middle|\; \exists d : a, b \in Q^d,\ bT^d \notin \tilde{Q}\right] \tag{2.2}$$

as a subring of $\Bbbk(Y) = \mathrm{Frac}(B)$. Let us assume that $x_i T \in \tilde{P}$ if and only if $i < t$. Then, $\tilde{x}_t := x_t$ defines $\tilde{H}_0 = E_P$ and the $\tilde{x}_i := x_i / x_t$ for $i < t$ define the remaining $t - 1$ components of $\tilde{H}$ passing through $\tilde{P}$. Note that for $i > t$, the $\tilde{x}_i := x_i / x_t$ are units. We know that $y_i T \in \tilde{Q}$ if and only if $i < t$. We define

$$\tilde{y}_i := \begin{cases} y_i / y_t & ; \quad i \neq t \\ y_i & ; \quad i = t \end{cases}$$

and claim that

$$\mathcal{O}_{\tilde{Y},\tilde{Q}} = \mathcal{O}_{\tilde{X},\tilde{P}}[\tilde{y}_1,\ldots,\tilde{y}_r] =: R. \tag{2.3}$$

Note that $\tilde{y}_i^n = x_i/x_t$ for $i > t$ is a unit and defines no component of $\tilde{H}$. Hence, once we have verified (2.3), we know that $\tilde{\pi}$ is a CBC. The inclusion "$\supseteq$" is obvious. To see "$\subseteq$", let $f = g/h \in \mathcal{O}_{\tilde{Y},\tilde{Q}}$ with $g, h \in B$. By (2.2), we can assume that $g, h \in Q$, so w.l.o.g. $g = y_i$. In fact, we may assume $g = y_t$ since

$$\frac{y_t}{h} \cdot \tilde{y}_i = \frac{y_t}{h} \cdot \frac{y_i}{y_t} = \frac{y_i}{h}.$$

Write $h = \sum_{i=1}^r h_i y_i$ and observe $f^{-1} = h_t + \sum_{i \neq t} h_i \tilde{y}_i \in R$. Since f is integral over $\mathcal{O}_{\tilde{X},\tilde{P}}$, there exist $\nu \in \mathbb{N}$ and certain $a_i \in \mathcal{O}_{\tilde{X},\tilde{P}} \subseteq R$ such that

$$f^\nu = a_0 \cdot f^{\nu-1} + \cdots + a_{\nu-2} \cdot f + a_{\nu-1}$$

Multiplication by $f^{1-\nu}$ yields

$$f = \sum_{i=0}^{\nu-1} a_i f^{-i} \in R.$$

Hence, we have verified that $\tilde{\pi}$ is a CBC.

By Corollary 1.32.(a), we note that $\mathrm{Rd}(\tilde{H})$ has less components than $\mathrm{Rd}(H)$. We can therefore repeat this process and eventually arrive at a situation as in (2.1), with parts (a) to (c) satisfied. Part (d) follows from Proposition 2.17. □

2.6 Global Kummer Coverings

Given a natural number $n \in \mathbb{N}$ not divisible by $\mathrm{char}(\mathbb{k})$ and an arrangement H inside a smooth variety X whose components have empty intersection, we will construct an n-fold CBC $Y \to X$ whose branch locus is H.

Definition 2.36. *Let $S := \mathbb{k}[x_0, \ldots, x_s]$ be the polynomial ring in $s+1$ variables and define $\varphi_n^\sharp : S \to S$ by $x_i \mapsto x_i^n$ for all i. This morphism of graded rings induces a morphism of projective varieties $\theta_n : \mathbb{P}^s \to \mathbb{P}^s$, which can be understood as the map $[a_0 : \ldots : a_s] \mapsto [a_0^n : \ldots : a_s^n]$. Although we will not consider this situation, for $n = \mathrm{char}(\mathbb{k})$, the morphism θ_n is the Frobenius morphism.*

Notation 2.37. *Let X be a scheme. We write $\mathcal{O}_X^{\ell+1} = \bigoplus_{i=0}^{\ell} \mathcal{O}_X e_i$. Whenever we consider an epimorphism $h : \mathcal{O}_X^{\ell+1} \twoheadrightarrow \mathscr{L}$ without further explanation, we mean that $\mathscr{L}$ is a globally generated line bundle and that the implicitly defined global sections*

$$h_i := h_X(e_i \cdot 1)$$

generate it, i.e. $\mathscr{L}_P$ *is generated by the stalks* $h_{i,P}$ *at every point* $P \in X$*. If* X *is a variety, denote by* $\phi_h : X \to \mathbb{P}^\ell$ *the corresponding morphism from* X *to projective space.*

Definition 2.38. *Let* X *be a* $\Bbbk$*-variety and* $h : \mathcal{O}_X^{\ell+1} \twoheadrightarrow \mathscr{L}$*. For any* $n \geq 2$*, we define the scheme* $X\left[\sqrt[n]{h}\right]$ *to be the fiber product*

$$\begin{tikzcd} X\left[\sqrt[n]{h}\right] \arrow[r, dotted, "\alpha"] \arrow[d, dotted, "\pi"'] \arrow[dr, phantom, "\times"] & \mathbb{P}^\ell \arrow[d, "\theta_n"] \\ X \arrow[r, "\phi_h"'] & \mathbb{P}^\ell \end{tikzcd}$$

together with the two canonical projection morphisms π *and* α*.*

Proposition 2.39. *Let* X *be a* $\Bbbk$*-variety and* $h : \mathcal{O}_X^{\ell+1} \twoheadrightarrow \mathscr{L}$*. Let* $\mathcal{I}_{ij}$ *be the homogeneous ideal sheaf of* $\mathcal{O}_X[T_0, \ldots, T_\ell]$ *which is locally generated by* $T_i^n h_j - T_j^n h_i$*. Here, we understand* h_i *as an element of* $\mathcal{O}_X(U)$ *under some local trivialization* $\mathscr{L}|_U \cong \mathcal{O}_U$*. Then, we write*

$$\mathcal{S} := \mathcal{O}_X\left[T_0, \ldots, T_\ell\right] \Big/ \textstyle\sum_{ij} \mathcal{I}_{ij}$$

and set $Y := \mathbb{P}\mathbf{roj}(\mathcal{S})$*. Then,* $X\left[\sqrt[n]{h}\right] \cong Y$ *and the canonical morphisms* π *and* α *are induced by* $\mathcal{O}_X \hookrightarrow \mathcal{S}$ *and* $\Bbbk[T_0, \ldots, T_\ell] \hookrightarrow \mathcal{S}$*, respectively.*

Remark. Note that the construction is independent on the local isomorphism $\mathscr{L}|_U \cong \mathcal{O}_U$ that is chosen: The elements h_i are defined up to (collective) multiplication by some $\alpha \in \mathcal{O}_X(U)^\times$ and $\alpha \cdot \mathcal{I}_{ij}(U) = \mathcal{I}_{ij}(U)$.

Proof. By the local nature of the fiber product, we may harmlessly assume that $X = \operatorname{Spec}(A)$ is affine and $\mathscr{L} \cong \mathcal{O}_U$. Without loss of generality, we may assume $h_0 = 1$ under this isomorphism since the h_i generate. The morphism ϕ_h is induced by

$$\begin{aligned} R := \Bbbk[T_0, \ldots, T_\ell] &\longrightarrow A \\ T_i &\longmapsto h_i \end{aligned}$$

Let $x_i := T_i/T_0$, and note that

$$\operatorname{im}(\phi_h) \subseteq D_*(T_0) = \Bbbk[x_1, \ldots, x_\ell] =: B.$$

By our assumption $h_0 = 1$, the corestriction $U \to D_*(T_0)$ is induced by the map $f : B \to A$ which sends x_i to h_i. Let $g : B \to B$ be the map $g(x_i) = x_i^n$.

Then, $U \times D_*(T_0) = \operatorname{Spec}(S)$ where

$$\begin{tikzcd}
S := A \otimes_B B & B \arrow[l, dotted] \\
A \arrow[u, dotted] & B \arrow[l, two heads, "f"] \arrow[u, "g"]
\end{tikzcd}$$

To show $Y \cong X\,[\sqrt[n]{h}]$, we have to prove $S \cong A[x_1, \dots, x_\ell]/I$, where I denotes the ideal $\left(x_i^n - h_i \;\middle|\; 1 \le i \le \ell \right)$. We choose S this way and check the universal property of the tensor product. Consider

$$\begin{tikzcd}
\tilde{S} & & \\
& S \arrow[ul, dotted, "t"] & B \arrow[l, "\bar{f}"] \arrow[ull, bend right, "\tilde{f}"'] \\
& A \arrow[u, "\bar{g}"] \arrow[uul, bend left, "\tilde{g}"] & B \arrow[l, two heads, "f"] \arrow[u, "g"]
\end{tikzcd}$$

where $\bar{f}(x_i) := h_i$ and $\bar{g}$ is canonical. Clearly, $\bar{f} \circ g = \bar{g} \circ f$. If $\tilde{g} \circ f = \tilde{f} \circ g$, we define a morphism $t : S \to \tilde{S}$ of A-algebras by $t(x_i) := \tilde{f}(x_i)$. It is easy to see that this is well-defined and uniqueness with respect to commutativity is also clear. $\square$

Notation 2.40. *If $f \in \mathscr{L}(X)$ is a global section of a line bundle, we denote by $\mathcal{Z}(f)$ the closed subscheme of X which is associated to the divisor of zeros of f.*

Definition 2.41. *Let X be a variety over the field $\Bbbk$. An **ℓ-building in** X is a globally generated line bundle $h : \mathcal{O}_X^{\ell+1} \twoheadrightarrow \mathscr{L}$ such that each $H_i := \mathcal{Z}(h_i)$ is irreducible and $H = H_0 + \cdots + H_\ell$ is an arrangement. We write $\mathcal{Z}(h) := H$.*

Remark 2.41.1. *Note that for all $i > 0$, the divisor $H_i - H_0$ is principal, i.e. we can write $H_i - H_0 = \operatorname{div}(x_i)$ for certain $x_i \in \Bbbk(X) =: K$. Assume that* $\operatorname{char}(\Bbbk)$ *does not divide n and consider the field*

$$L := K\,[\sqrt[n]{x_1}, \dots, \sqrt[n]{x_\ell}]\,,$$

which is a Kummer extension of K. If we denote by P_i the generic point of H_i, we also have valuations $v_i : K \to \mathbb{Z} \cup \{\infty\}$ corresponding to the discrete valuation rings $\mathcal{O}_{X,P_i}$, satisfying $v_i(x_j) = \delta_{ij}$ (the Kronecker delta[3]).

*If we set $Y := X\,[\sqrt[n]{h}]$, the morphism $\pi : Y \to X$ will turn out to be an n-fold CBC, which we refer to as the **global Kummer covering** associated to h.*

[3] This means $\delta_{ii} = 1$ and $\delta_{ij} = 0$ for $i \neq j$.

Example 2.41.2. *Consider the case where $X \subseteq \mathbb{P}^s$ is a projective variety with coordinate ring S. Let $\mathscr{L} = \mathcal{O}_X(1)$. A set of linear forms*

$$h = \{ h_0, \ldots, h_\ell \} \subset \mathscr{L}(X) = S_1$$

then defines $\ell + 1$ hyperplanes $H_i = Z_*(h_i)$ forming an ℓ-building. For $s = 1$, this is a set of points and for $s = 2$, it is a set of projective lines.

Note that $H_0 \cap \ldots \cap H_\ell = \emptyset$ if and only if h is a set of generators. This is equivalent to requiring that the geometric dual $\{ H_0^*, \ldots, H_\ell^* \}$ is not completely contained in any hyperplane.

Remark 2.41.3. *If h is an ℓ-building and $P \in X$ any point, then there exists some index j such that under $\mathscr{L}(X) \to \mathscr{L}_P \xrightarrow{\sim} \mathcal{O}_{X,P} \twoheadrightarrow \Bbbk$, the image of h_j is nonzero. In other words, $h_j(P) \neq 0$. Indeed, this is what it means for $\mathscr{L}$ to be globally generated by the h_i.*

Scenario 2.42. *Let X be a nonsingular $\Bbbk$-variety, $n \in \mathbb{N}$ not divisible by* $\operatorname{char}(\Bbbk)$ *and $h : \mathcal{O}_X^{\ell+1} \twoheadrightarrow \mathscr{L}$ an ℓ-building. Let*

$$\pi : Y := X\left[\sqrt[n]{h}\right] \longrightarrow X.$$

We write $H_i := \mathcal{Z}(h_i)$ and $H := H_0 + \cdots + H_\ell$. We set $K := \Bbbk(X)$ and after Corollary 2.46, also $L := \Bbbk(Y)$.

Proposition 2.43. *In Scenario 2.42, let $U = \operatorname{Spec}(A) \subseteq X$ be an open subset where $\mathscr{L}$ is trivial and $h_v \in A^\times$ for some v. With $x_i := h_i/h_v$, we then have*

$$\pi^{-1}(U) \cong \operatorname{Spec}\left(A\left[y_0, \ldots, y_\ell\right]\right) \text{ for } y_i \in \sqrt[n]{x_i}$$

In particular, by Remark 2.41.3, π is a finite morphism.

Proof. We may assume that $h_v = 1 \in A$ since everything is independent of the choice of the local isomorphism $\mathscr{L}(U) \cong \mathcal{O}_X(U)$. Without loss of generality, we assume $v = 0$. We are then considering the ring $R = A[z_0, \ldots, z_\ell]$ where $z_i^n h_j = h_i z_j^n$ for all i and j. If $P \in \operatorname{Proj}(R)$, then there exists some i such that $z_i \notin P$. Since $z_i^n = h_i z_0^n$, we know $z_0 \notin P$. Thus, $D_*(z_0) = \operatorname{Proj}(R) = \pi^{-1}(U)$ and

$$\operatorname{Spec}\left(A\left[y_0, \ldots, y_\ell\right]\right) = \operatorname{Spec}\left(A\left[\tfrac{z_0}{z_0}, \ldots, \tfrac{z_\ell}{z_0}\right]\right) = \operatorname{Spec}\left((R_{z_0})_0\right) \cong D_*(z_0). \qquad \square$$

Corollary 2.44. *In Scenario 2.42, $\mathcal{O}_{Y,Q} = \mathcal{O}_{X,P}\left[y_0, \ldots, y_\ell\right]$ for a point $Q \in Y$ and $P := \pi(Q)$. Furthermore, $y_i \in \mathfrak{m}_Q$ if and only if $h_i \in \mathfrak{m}_P$. The morphism π is an n-fold CBC.*

Proof. Assume $h_v \in \mathcal{O}^{\times}_{X,P}$. Since $h_i = h_v y_i^n$ and $\mathfrak{m}_Q$ is prime, we can immediately see $y_i \in \mathfrak{m}_Q \Leftrightarrow h_i \in \mathfrak{m}_Q \cap \mathcal{O}_{X,P} = \mathfrak{m}_P$. □

Lemma 2.45. *Let K be a field containing all n-th roots of unity. Assume that there are $x_1, \dots, x_\ell \in K$ and valuations $v_i : K \to \mathbb{Z} \cup \{\infty\}$ with $v_i(x_j) = \delta_{ij}$. Then, $L := K[y_1, \dots, y_\ell]$ is a field for any $y_i \in \sqrt[n]{x_i}$ and the Galois group of L over K is isomorphic to $\mathbb{Z}_n^\ell = (\mathbb{Z}/(n))^\ell$. Consequently, L has degree n^ℓ over K.*

Proof. We use the notation $K^{\times n} = \{ x^n \mid x \in K^\times \}$. Let C be the subgroup of $K^\times$ generated by the x_i and $K^{\times n}$. The valuations v_i can be understood as a map $\varphi : C \to \mathbb{Z}^\ell$ and the composition

$$C \xrightarrow{\varphi} \mathbb{Z}^n \twoheadrightarrow \mathbb{Z}_n^\ell$$

clearly has kernel $K^{\times n}$. Thus, we can conclude $C/K^{\times n} \cong \mathbb{Z}_n^\ell$ and apply the well-known result [Bos, Kapitel 4.9, Satz 1 und Lemma 2] from Kummer theory. □

Corollary 2.46. *If h is an ℓ-building inside a variety X, then $X[\sqrt[n]{h}]$ is a variety.*

Proof. Let $Q \in Y$ and $P := \pi(Q)$. Then, $K := \mathrm{Frac}(\mathcal{O}_{X,P}) = \Bbbk(X)$. By Remark 2.41.1 and Lemma 2.45, the field $L := K[y_1, \dots, y_\ell]$ is a Galois extension of K. Since $\mathcal{O}_{Y,Q} = \mathcal{O}_{X,P}[y_1, \dots, y_\ell]$ is a subring of L, it must be an integral domain. □

Corollary 2.47. *In Scenario 2.42, L is a Galois extension of K. It has degree n^ℓ and Galois group $\mathbb{Z}_n^\ell$.* □

$$\begin{tikzcd}
X(\sqrt[n]{h}) \arrow[rr, "\tilde{\beta}"] \arrow[dd, "\tilde{\pi}"'] \arrow[ddrr, dashed, "\gamma"] & & X[\sqrt[n]{h}] \arrow[dd, "\pi"] \arrow[dl, phantom, "\circlearrowleft"] \\
 & \circlearrowleft & \\
\tilde{X} \arrow[rr, "\beta"] & & X
\end{tikzcd}$$

Figure 2.3: Kummer Covering

Definition 2.48. *Let X be a nonsingular $\Bbbk$-variety and h an ℓ-building in X. With notation as in Theorem 2.35, we let $\tilde{\pi} : X(\sqrt[n]{h}) \to \tilde{X}$ denote the regularization of $\pi : X[\sqrt[n]{h}] \to X$. We obtain an induced map $\gamma : X(\sqrt[n]{h}) \to X$. See also Figure 2.3.*

Chapter 3

Line Arrangements

In this section, we apply the results from Chapter 2 to a particular scenario. This is largely based on the paper [Hir] and the book [BHH] where Hirzebruch also develops the theory of Chapter 2 in the special case of surfaces.

Ultimately, we will prove the key argument used in Proposition 1.10, completing the proof of Theorem 1.16.

Scenario 3.1. *We are working over the field $\Bbbk = \mathbb{C}$. Let $n \geq 2$, $X := \mathbb{P}^2$ and consider an arrangement $H = H_0 + \cdots + H_\ell$ of projective lines, i.e. $H_i = \mathcal{Z}(h_i)$ for certain linear, homogeneous polynomials $h_i \in \mathbb{C}[x_0, x_1, x_2]_1$. Assume that the H_i have empty intersection, hence we may understand $h : \mathcal{O}_X^{\ell+1} \twoheadrightarrow \mathcal{O}_X(1)$ as a way to globally generate the twisting sheaf. We simply write t_r instead of $t_r(2, H)$, the number of points in the plane where r of the lines intersect. We define*

$$m := (\ell + 1) \qquad f_0 := \sum_{r \geq 2} t_r \qquad f_1 := \sum_{r \geq 2} r \cdot t_r \tag{3.1}$$

We set $Y := X[\sqrt[n]{h}]$ and $\tilde{Y} := X(\sqrt[n]{h}) \to \tilde{X}$. We will denote morphisms as in Figure 2.3. Note that β is the blow-up of X in all r-points for $r > 2$ and $\tilde{\beta}$ the blow-up of Y in the points that lie above those. Let

$$N := \sum_{r \geq 3} t_r = f_0 - t_2$$

be the number of the redundant points $P_1, \ldots, P_N \in X$. Let $r_i := r_H(P_i)$. We also denote the branch locus if $\tilde{\pi}$ by $\tilde{H} := \tilde{H}_0 + \cdots + \tilde{H}_{\ell+N}$, where

$$\tilde{H}_i = \begin{cases} \beta^\intercal(H_i) & ; \quad i \leq \ell \\ \beta^{-1}(P_{i-\ell}) & ; \quad i > \ell \end{cases}$$

We define $\tilde{t}_r := t_r(2, \tilde{H})$. By Theorem 2.35, we know that $\tilde{t}_r = 0$ for all $r > 2$. We also write $c_i := c_i(\mathcal{T}_{\tilde{Y}})$ for the i-th Chern class of the tangent sheaf of $\tilde{Y}$.

The contents of Sections 3.1 and 3.2 are straightforward calculations. Section 3.3 assembles these pieces to prove Theorem 3.21.

3.1 Euler Characteristic

We will calculate the Euler characteristic of the complex surface $\tilde{Y}$. This number is also the degree of the top Chern class c_2, as we will see later in Theorem 3.18.

Fact 3.2. *The Euler characteristic of complex projective space is* $\chi(\mathbb{P}^n_{\mathbb{C}}) = n + 1$.

Proof. The fact that $\mathbb{P}^n_{\mathbb{C}} = E_0 \cup \cdots \cup E_{2n}$ has a cellular decomposition with $\dim(E_d) = d$ is well known, see [Hat, Example 0.6] for instance. Then, [Hat, Theorem 2.44] immediately implies our claim. $\square$

Lemma 3.3. *In Scenario 3.1, for any $P \in X$, the exceptional divisor $E_P = \beta^{-1}(P)$ is isomorphic to the projective line $\mathbb{P}^1$.*

Proof. We may choose an affine neighborhood $U = \operatorname{Spec}(\Bbbk[x, y])$ of P where it is the origin, i.e. the maximal ideal

$$P = (x, y) \subset \Bbbk[x, y].$$

Then, we know that $E_P = \beta^{-1}(P)$ corresponds to the homogeneous ideal $\bigoplus_{d \geq 0} P^{d+1} T^d$ inside the blow-up algebra $\Bbbk[x, y][PT]$. Since

$$\bigoplus_{d \geq 0} P^d \big/ P^{d+1} = \bigoplus_{d \geq 0} \Bbbk[x, y]_d,$$

the homogeneous coordinate ring of E_P is $\Bbbk[x, y]$, hence $E_P \cong \mathbb{P}^1$. $\square$

Lemma 3.4. *In Scenario 3.1,* $\tilde{t}_2 = f_1 - t_2$.

Proof. Note that the strict transform $\tilde{H}_i$ of any line H_i passing through an r-point P will intersect with $E_P = \beta^{-1}(P)$. Hence, E_P intersects with $\tilde{H}_i$ if and only if $i \in \lambda(P)$. Thus,

$$\tilde{t}_2 = t_2 + \sum_{i=1}^{N} |\lambda(P_i)| = t_2 + \sum_{i=1}^{N} r(P_i) = 1 \cdot t_2 + \sum_{r \geq 3} r \cdot t_r = f_1 - t_2.$$ $\square$

Lemma 3.5. *If $H = H_0 + \cdots + H_\ell$ is any arrangement inside a (nonsingular) surface X, then with $t_r := t_r(2, H)$,*

$$\chi(H) = \sum_{i=0}^{\ell} \chi(H_i) - \sum_{r \geq 2} (r - 1) \cdot t_r.$$

Proof. Let $Z_i \subset H_i$ be the (finite) set of points where H_i intersects with some other part of the arrangement. Let $Z := \bigcup_{i=0}^{\ell} Z_i$ and $Z' := \dot{\bigcup}_{i=0}^{\ell} Z_i$. Clearly,

$$|Z| = \sum_{r \geq 2} t_r \qquad\qquad |Z'| = \sum_{r \geq 2} r t_r \tag{3.2}$$

since in the disjoint union Z', each point $P \in Z$ is counted exactly $r(P)$ times. Hence by Proposition 2.23,

$$\chi(H) = \sum_{i=0}^{\ell} \chi(H_i \setminus Z_i) + \sum_{P \in Z} \chi(P) = \sum_{i=0}^{\ell} \chi(H_i) - \chi(Z') + \chi(Z)$$

yields the desired result by substituting (3.2). $\square$

Proposition 3.6. *In Scenario 3.1, the Euler characteristic of $\tilde{Y}$ can be calculated as*

$$n^{2-\ell} \cdot \chi(\tilde{Y}) = n^2 \cdot (3 - 2m + f_1 - f_0) + 2n \cdot (m - f_1 + f_0) + (f_1 - t_2).$$

Proof. The morphism $\tilde{\pi} : \tilde{Y} \to \tilde{X}$ is an RCBC by Theorem 2.35 and of degree n^ℓ by Corollary 2.47. Hence by Corollary 2.24,

$$n^{2-\ell} \cdot \chi(\tilde{Y}) = n^2 \cdot \chi(\tilde{X} \setminus \tilde{H}) + n \cdot \chi(\tilde{H} \setminus \mathrm{Rd}\,(\tilde{H})) + \chi(\mathrm{Rd}\,(\tilde{H})).$$

We analyze the coefficients on the right hand side. With the isomorphisms $H_i \cong \mathbb{P}^1$ and $\tilde{X} \setminus \tilde{H} \cong X \setminus H$, Lemma 3.5 yields

$$\begin{aligned}
\chi(\tilde{X} \setminus \tilde{H}) &= \chi(X) - \chi(H) \\
&= \chi(\mathbb{P}^2) - m \cdot \chi(\mathbb{P}^1) + f_1 - f_0 \\
&= 3 - 2m + f_1 - f_0.
\end{aligned}$$

Since $\tilde{H}$ is strict and by Lemma 3.4, the constant term is easily calculated as

$$\chi(\mathrm{Rd}(\tilde{H})) = \tilde{t}_2 = f_1 - t_2.$$

We now turn to the linear coefficient. We know $\tilde{t}_r = 0$ for $r > 2$. Furthermore, for all $0 \leq i \leq \ell + N$, we have $\tilde{H}_i \cong \mathbb{P}^1$ by Lemma 3.3. Thus, Lemma 3.5 implies

$$\begin{aligned}
\chi(\tilde{H} \setminus \mathrm{Rd}(\tilde{H})) &= \chi(\tilde{H}) - \chi(\mathrm{Rd}(\tilde{H})) \\
&= ((m + N) \cdot \chi(\mathbb{P}^1) - \tilde{t}_2) - \tilde{t}_2 \\
&= 2(m + N - \tilde{t}_2) \\
&= 2(m + f_0 - f_1).
\end{aligned}$$

$\square$

3.2 The Canonical Divisor

We now calculate the self-intersection number of a canonical divisor on $\tilde{Y}$. This number is the degree of the square $c_1^2(\tilde{Y})$ of the first Chern class, as we will see in Proposition 3.17.

Notation 3.7. *Let X be a surface and H a divisor. In order to deobfuscate the notation, we will write H to refer to the class $[H] \in A_1(X)$.*

Furthermore, for any $\alpha \in A_0(X)$, we will simply write α instead of $\int_X \alpha$. For example, the term H^2 now means $\int_X [H]^2$.

Theorem 3.8 (Adjunction Formula). *If C is a nonsingular curve of genus g on a surface X, then*

$$2g - 2 = C(C + K_X).$$

Proof. This is precisely [Har, Proposition V.1.5]. □

Fact 3.9. *If C is a nonsingular, complex curve of genus g, then $\chi(C) = 2 - 2g$.*

Proof. C has a cellular decomposition with $2g$ cells in dimension one and one cell in each of the dimensions zero and two, as explained in [Hat, Cell Complexes, Chapter 0]. Thus, we are done by [Hat, Theorem 2.44]. □

Proposition 3.10. *Let $\pi : Y \to X$ be an n-fold RCBC of complex surfaces with branch locus H. Denote by $\bar{H}$ the disjoint union of its components. Then,*

$$\tfrac{n^2}{\deg \pi} \cdot K_Y^2 = n^2 \cdot (K_X^2 + K_X H + T) - 2n \cdot T + (T - K_X H). \tag{3.3}$$

where $T := 2 \cdot t_2(2, H) - \chi(\bar{H})$.

Proof. Let $H = H_0 \cup \cdots \cup H_\ell$ be the irreducible components. By Theorem 3.8 and Fact 3.9,

$$-\chi(H_i) = H_i^2 + H_i K_X$$

for each i. Also, $2 \cdot t_2(2, H) = \sum_{i=0}^{\ell} \sum_{j \neq i} H_i H_j$. We conclude

$$\begin{aligned} T = 2 \cdot t_2(2,H) - \sum_{i=0}^{\ell} \chi(H_i) &= \sum_{i=0}^{\ell} \left(\textstyle\sum_{j \neq i} H_i H_j + H_i^2 + H_i K_X \right) \\ &= H^2 + K_X H. \end{aligned}$$

By Corollary 2.29,

$$\tfrac{1}{\deg \pi} \cdot K_Y^2 = \left(K_X + \tfrac{n-1}{n} \cdot H \right)^2 = K_X^2 + \tfrac{2n-2}{n} \cdot K_X H + \tfrac{n^2+1-2n}{n^2} \cdot H^2.$$

Substituting H^2 by $T - K_X H$, we obtain

$$\tfrac{n^2}{\deg \pi} \cdot K_Y^2 = n^2 \cdot K_X^2 + 2(n^2 - n) \cdot K_X H + (n^2 + 1 - 2n) \cdot (T - K_X H). \quad \square$$

Lemma 3.11. *In Scenario 3.1, the following equations hold:*

(a). $K_{\tilde{X}} = \beta^*(K_X) + \sum_{i=1}^{N} \tilde{H}_{i+\ell}$

(b). $\tilde{H} = \beta^*(H) - \sum_{i=1}^{N}(r_i - 1)\tilde{H}_{i+\ell}$

Metaproof. These are [Har, Propositions V.3.3 and V.3.6]. □

Lemma 3.12. *In Scenario 3.1,*

$$\tilde{H}_i\tilde{H}_j = \begin{cases} H_iH_j & ; \quad i,j \le \ell \\ -\delta_{ij} & ; \quad \textit{otherwise} \end{cases}$$

Metaproof. This is the content of [Har, Proposition V.3.2]. □

Proposition 3.13. *In Scenario 3.1, we have*

$$n^{2-\ell} \cdot K_{\tilde{Y}}^2 = n^2(9 + 3f_1 - 4f_0 - 5m) + 4n(m - f_1 + f_0) + (f_1 - f_0 + t_2 + m)$$

Proof. By Lemmata 3.11 and 3.12 and Proposition 2.28,

$$\begin{aligned} K_{\tilde{X}}^2 &= K_X^2 - N \\ &= K_X^2 - f_0 + t_2 \end{aligned} \qquad \text{and} \qquad \begin{aligned} K_{\tilde{X}}\tilde{H} &= K_X H + \sum_{r\ge 3}(r-1)t_r \\ &= K_X H + f_1 - f_0 - t_2. \end{aligned}$$

We can choose the canonical divisor of $X = \mathbb{P}^2$ as $K_X = -3L$ for any line $L \subset X = \mathbb{P}^2$. For instance, we may choose $L = H_i$ for all i. This is well-known, see [Har, Examples II.8.20.3, V.1.4.2 and V.1.4.4] for instance. Thus, $K_X^2 = 9$ and $K_X H = -3m$. We conclude

$$K_{\tilde{X}}^2 = 9 - f_0 + t_2 \qquad \text{and} \qquad K_{\tilde{X}}\tilde{H} = f_1 - f_0 - t_2 - 3m.$$

Substituting for these values in Proposition 3.10, we remark that

$$\begin{aligned} T &= 2\tilde{t}_2 - 2(N + \ell + 1) = 2(f_1 - t_2 - N - m) \\ &= 2(f_1 - t_2 - f_0 + t_2 - m) = 2(f_1 - f_0 - m) \end{aligned}$$

and calculate

$$\begin{aligned} n^{2-\ell} \cdot K_{\tilde{Y}}^2 &= n^2((9 - f_0 + t_2) + (f_1 - f_0 - t_2 - 3m) + T) \\ &\quad - 2nT + (T - f_1 + f_0 + t_2 + 3m) \\ &= n^2(9 - 2f_0 + f_1 - 3m + 2(f_1 - f_0 - m)) \\ &\quad - 4n(f_1 - f_0 - m) \\ &\quad + (2(f_1 - f_0 - m) - f_1 + f_0 + t_2 + 3m) \\ &= n^2(9 + 3f_1 - 4f_0 - 5m) \\ &\quad + 4n(m - f_1 + f_0) + (f_1 - f_0 + t_2 + m) \end{aligned}$$

□

3.3 The Miyaoka-Yau Inequality

The Miyaoka-Yau inequality relates the Chern numbers of complex surfaces. It was proved independently by Shing-Tung Yau and Yoichi Miyaoka in 1977. We quote the latter result [Miy1, Theorem 4]:

Theorem 3.14 (The Miyaoka-Yau Inequality). *Let X be a nonsingular, complex, projective surface of general type. Let $c_i := c_i(\mathcal{T}_X)$ be the corresponding i-th Chern class. Then,*

$$c_1^2 \leq 3 \cdot c_2 \tag{3.4}$$

where Notation 3.7 applies.

Van de Ven (1966) and Fedor Bogomolov (1978) proved weaker versions with the constant 3 replaced by 8 and 4, respectively. Hirzebruch showed that Theorem 3.14 is best possible, by finding infinitely many examples where equality holds. Hirzebruch constructed these examples as Kummer coverings of the projective plane. Theorem 3.21, a byproduct of these efforts, was used by Kelly to prove the complex Sylvester-Gallai Theorem. To get there, we will need a corollary of Theorem 3.14.

Definition 3.15. *Let X be a projective variety. Then,*

$$R(X) := \bigoplus_{d \geq 0} \mathcal{H}^0\left(X, \omega_X^{\otimes d}\right)$$

*is called the **canonical ring of** X. The **Kodaira dimension of** X is defined to be the transcendence degree of R over $\Bbbk$, i.e.*

$$\mathrm{kod}(X) := \mathrm{tr.\,deg}_{\Bbbk}(R(X)) - 1.$$

Sometimes, the notation $\mathrm{kod}(X) = -\infty$ is used for the cases where our definition yields $\mathrm{kod}(X) = -1$.

Corollary 3.16. *If X is a nonsingular, complex, projective surface with an effective canonical divisor K_X and $K_X^2 > 0$, then it satisfies (3.4).*

Proof. If a projective variety X has an effective canonical divisor, then ω_X has a global section. This follows from [Har, Proposition II.7.7], for instance. Hence, $\mathrm{kod}(X) \geq 0$ and by the Enriques-Kodaira classification of complex surfaces [BHPvdV, Chapter VI, Theorem 1.1], the condition $K_X^2 > 0$ means that X is of general type. Hence, we can apply Theorem 3.14. □

Let us now verify that Euler characteristic and self-intersection number of a canonical divisor correspond to c_2 and c_1^2, respectively.

Proposition 3.17. *Let X be a smooth, projective variety. Then, $c_1(\mathcal{T}_X) = -[K_X]$.*

Proof. Set $s := \dim(X)$. Let us write $c_T(\Omega_X) = \prod_{i=1}^{s}(1 + a_i T)$ for formal variables a_i. By property C5, $c_T(\omega_X) = c_T(\bigwedge^s \Omega_X) = 1 + (a_1 + \cdots + a_s)T$. Together with property C1, this means $[K_X] = c_1(\omega_X) = c_1(\Omega_X)$. Again using property C5, we calculate $c_1(\mathcal{T}_X) = c_1(\Omega_X^\vee) = -c_1(\Omega_X) = -[K_X]$. □

Theorem 3.18 (Gauss-Bonnet-Formula). *Let X be a nonsingular, complex, projective variety of dimension s. Then,*

$$\int_X c_s(\mathcal{T}_X) = \chi(X^{\mathrm{an}}).$$

Metaproof. Read the discussion following [Huy, Corollary 5.1.4]. □

Proof. We give an alternative proof requiring slightly less complex geometry. Write $\Omega_X^p := \bigwedge^p \Omega_X$ for the p-th exterior power of Ω_X. We will use the *Borel-Serre-Identity*, given in [Ful1, Example 3.2.5]:

$$\sum_{p=0}^{s} (-1)^p \cdot \mathrm{ch}\left(\Omega_X^p\right) \cdot \mathrm{td}(\mathcal{T}_X) = c_s(\mathcal{T}_X). \tag{3.5}$$

Note that s is the rank of Ω_X and $\Omega_X^\vee = \mathcal{T}_X$ by definition. As a second tool, we require the Hirzebruch-Riemann-Roch Theorem (Theorem 1.46) to conclude

$$\int_X \mathrm{ch}\left(\Omega_X^p\right) \cdot \mathrm{td}(\mathcal{T}_X) = \chi\left(X, \Omega_X^p\right). \tag{3.6}$$

Finally, we require the Hodge Decomposition Theorem, which we quote from [Huy, Corollaries 2.6.21 and 3.2.12] as

$$H^r(X^{\mathrm{an}}, \mathbb{C}) = \bigoplus_{p+q=r} \mathcal{H}^q\left(X, \Omega_X^p\right). \tag{3.7}$$

Putting it all together, we obtain

$$\begin{aligned}
\int_X c_s(\mathcal{T}_X) &= \sum_{p=0}^{s} (-1)^p \cdot \int_X \mathrm{ch}\left(\Omega_X^p\right) \cdot \mathrm{td}(\mathcal{T}_X) && \text{by (3.5)} \\
&= \sum_{p=0}^{s} (-1)^p \cdot \chi\left(X, \Omega_X^p\right) && \text{by (3.6)} \\
&= \sum_{p,q} (-1)^{p+q} \cdot \mathrm{rank}\left(\mathcal{H}^q\left(X, \Omega_X^p\right)\right) \\
&= \sum_{r=0}^{s} (-1)^r \cdot H^r(X^{\mathrm{an}}, \mathbb{C}) && \text{by (3.7)} \\
&= \chi(X^{\mathrm{an}}).
\end{aligned}$$

□

We are now ready to begin proving Theorem 3.21, which will complete the proof of Theorem 1.16.

Lemma 3.19. *In Scenario 3.1, the polynomial*

$$F(T) := T^2 \cdot (f_0 - m) + 2T \cdot (2m - f_1) + 4(f_1 - t_2 - m) \tag{3.8}$$

satisfies $F(n+1) = n^{2-\ell} \cdot (3c_2 - c_1^2)$.

Proof. Using Propositions 3.6 and 3.13,

$$\begin{aligned} n^{2-\ell} \cdot (3c_2 - c_1^2) &= n^2 \cdot (9 - 6m + 3f_1 - 3f_0 - 9 - 3f_1 + 4f_0 + 5m) \\ &\quad + 2n \cdot (3m - 3f_1 + 3f_0 - 2m + 2f_1 - 2f_0) \\ &\quad + (3f_1 - 3t_2 - f_1 + f_0 - t_2 - m) \\ &= n^2 \cdot (f_0 - m) + 2n \cdot (m + f_0 - f_1) \\ &\quad + (2f_1 - 4t_2 + f_0 - m), \end{aligned}$$

and substituting $T - 1$ for n yields

$$\begin{aligned} &= (T-1)^2(f_0 - m) + 2(T-1)(m + f_0 - f_1) \\ &\quad + (2f_1 - 4t_2 + f_0 - m) \\ &= T^2(f_0 - m) + 2T(m + f_0 - f_1 - f_0 + m) \\ &\quad + (2f_1 - 4t_2 + f_0 - m) + (f_0 - m) - 2(m + f_0 - f_1), \end{aligned}$$

which is precisely (3.8). □

Lemma 3.20. *In Scenario 3.1,* $\frac{m(m-1)}{2} = \sum_{r=2}^{m} \frac{r(r-1)}{2} \cdot t_r$.

Proof. This follows because any two projective lines intersect in precisely one point and we have $\binom{m}{2} = \frac{m(m-1)}{2}$ choices for such a pair. Equivalently, we may count all pairs of lines that intersect in an r-point, for each r. □

Theorem 3.21. *Let* $\{ H_0, \dots, H_\ell \}$ *be a set of projective lines in the complex projective plane* $\mathbb{P}^2_{\mathbb{C}}$ *and* $H = H_0 + \cdots + H_\ell$ *the corresponding divisor. Set* $t_r := t_r(2, H)$. *If* $t_2 = t_\ell = t_{\ell+1} = 0$, *then* $t_3 > 0$.

Proof. We are in Scenario 3.1. First note that we may assume $\ell \geq 4$ for obvious reasons (otherwise, none of the lines can intersect by assumption). For $\ell = 4$, Lemma 3.20 yields $10 = 3 \cdot t_3$, so we may even assume $\ell \geq 5$. Let

$$H' := H_0 + \cdots + H_5$$

and set $t'_r := t_r(2, H')$. By possibly re-ordering the H_i, we may assume $t'_r = 0$ for $r > 3$. Indeed, if such a choice was impossible, then all but two lines of H would have to intersect in a common point, quickly yielding the contradiction $t_2 \neq 0$. Since $-H' = 2K_X$, we can describe a bicanonical divisor on $\tilde{X}$ as

$$2K_{\tilde{X}} = \beta^*(2K_X) + 2E = -\beta^*(H') + 2E \tag{3.9}$$

where $E = \sum_{j=1}^{N} \tilde{H}_{j+\ell}$ is the exceptional divisor of the blow-up β. Let

$$\beta^*(H') = \sum_{i=0}^{5} \tilde{H}_j + \sum_{j=1}^{N} v_j \tilde{H}_{j+\ell} \tag{3.10}$$

where $0 \leq v_j \leq 3$ by our earlier assumption. Combining (3.9) and (3.10), we may write the bicanonical divisor on $\tilde{X}$ as

$$2K_{\tilde{X}} = -\sum_{i=0}^{5} \tilde{H}_j + \sum_{j=1}^{N} (2 - v_j)\tilde{H}_{j+\ell}$$

By Corollary 2.27, we conclude that

$$\begin{aligned}
2nK_{\tilde{Y}} &= \tilde{\pi}^*\left(2nK_{\tilde{X}} + 2(n-1)\tilde{H}\right) \\
&= \tilde{\pi}^*\left(\sum_{i=0}^{5} -n\tilde{H}_i + \sum_{i=0}^{\ell}(2n-2)\tilde{H}_i + \sum_{j=1}^{N}\left(n(2-v_j) + 2(n-1)\right)\tilde{H}_{j+\ell}\right) \\
&= \tilde{\pi}^*\left(\sum_{i=0}^{5}(n-2)\tilde{H}_i + \sum_{i=6}^{\ell}(2n-2)\tilde{H}_i + \sum_{j=1}^{N}((4-v_j)n-2)\tilde{H}_{j+\ell}\right) \\
&=: \tilde{\pi}^*(D)
\end{aligned}$$

must be effective for $n \geq 2$. Unless $t_3 > 0$, we may assume

$$\frac{m(m-1)}{2} = \sum_{r=2}^{m} \frac{r(r-1)}{2} \cdot t_r \geq \sum_{r=2}^{\ell+1} 6t_r = 6N. \tag{3.11}$$

by Lemma 3.20. Note that $\sum_{j=1}^{N} v_i = \binom{6}{2} = 15$. For $n \geq 2$, Lemma 3.12 then yields

$$\begin{aligned}
D^2 &= \left(\sum_{i=0}^{5}(n-2)\tilde{H}_i + \sum_{i=6}^{\ell}(2n-2)\tilde{H}_i + \sum_{j=1}^{N}((4-v_j)n-2)\tilde{H}_{j+\ell}\right)^2 \\
&= 36(n-2)^2 + 12(m-6)(n-2)(2n-2) + (2n-2)^2(m-6)^2 \\
&\quad - (4n-2)N + 15 \\
&\geq 36(n-2)^2 + 12(m-6)(n-2)(2n-2) + (2n-2)^2(m-6)^2 \\
&\quad - (4n-2) \cdot \frac{m(m-1)}{12} + 15 \\
&=: \delta(n, m)
\end{aligned}$$

For $n \geq 3$ or $m \geq 8$, it is easy to see that $\delta(n, m) > 0$ and thus, $K^2_{\tilde{Y}} > 0$ in this case. See also Figure 3.1. Also see Figure 3.2 for a visual graph of δ.

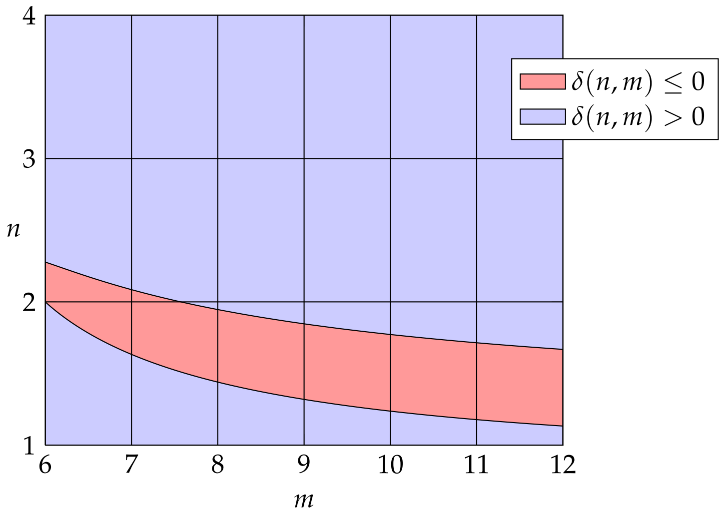

Figure 3.1: In blue: The area where $\delta(n, m) > 0$.

By Corollary 3.16, $\tilde{Y}$ satisfies the Miyaoka-Yau inequality for $n = 3$ and the polynomial $F(T)$ from Lemma 3.19 yields

$$\begin{aligned} 0 \leq F(4) &= 16(f_0 - m) + 8(2m - f_1) + 4(f_1 - t_2 - m) \\ &= 16f_0 - 8f_1 + 4f_1 - 4t_2 - 4m \\ &= 16f_0 - 4f_1 - 4m \\ &= 4 \cdot \sum_{r \geq 2} (4 - r)t_r - 4m. \end{aligned}$$

As an immediate result, $t_3 \geq m + \sum_{r \geq 4} (4 - r)t_r > 0$. $\square$

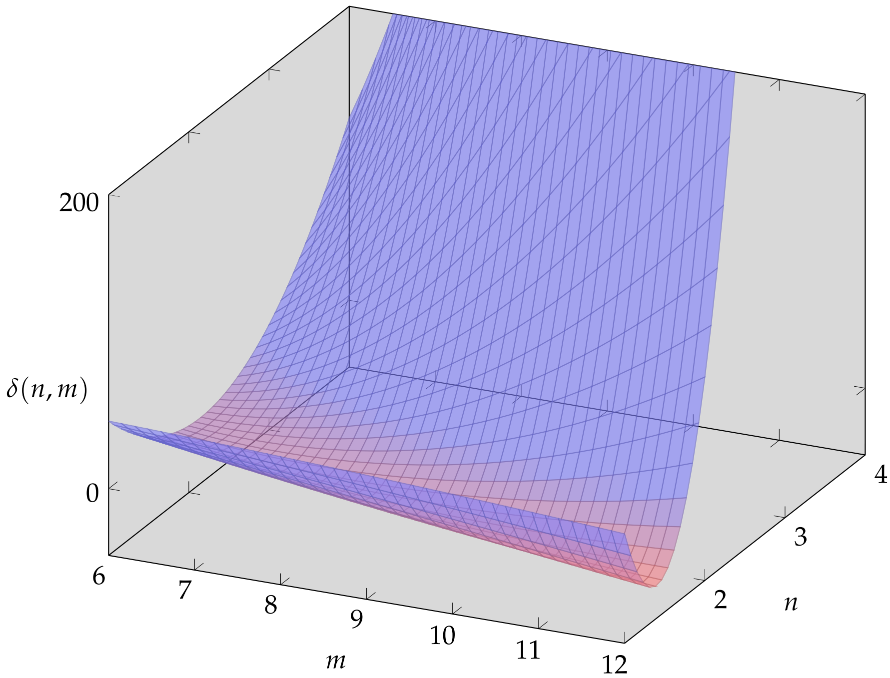

Figure 3.2: Visualization of δ.

Chapter 4

Perspectives

Since the ultimate goal is a resolution of Conjecture 1.9, we naturally wonder about the case $k > 2$. In Section 4.1, we give some pointers on how to tackle it by means of CBCs. One might also wonder how these results carry on to the world of positive characteristic, so we discuss possible prospects in Section 4.2.

4.1 Approaches to the case $k > 2$

There had been no motivation for Hirzebruch to study CBCs in higher dimensions because his motivation was classification of surfaces. Consequently, Kelly only had results in dimension two at his disposal. His proof therefore required a clever geometric trick, namely Proposition 1.14. Now, if we want to prove Conjecture 1.9 for $k > 2$, we will most likely have to leave the comfortable Scenario 3.1 and move to

Scenario 4.1. *Let $X := \mathbb{P}^s_{\mathbb{C}}$ and $h : \mathcal{O}_X^{\ell+1} \twoheadrightarrow \mathcal{O}_X(1)$ an ℓ-building of hyperplanes. For $n \geq 2$, we set $Y := X[\sqrt[n]{h}]$ and $\tilde{Y} := X(\sqrt[n]{h}) \to \tilde{X}$. Write $H_i := \mathcal{Z}(h_i)$ and $H := H_0 + \cdots + H_\ell$.*

Again, we want to study numerical invariants of the variety $\tilde{Y}$, express them by means of the combinatorial data of H and use relations between these invariants to infer Conjecture 1.9.

Fortunately, the Miyaoka-Yau Inequality (Theorem 3.14) is just the tip of an iceberg of inequalities involving Chern classes of complex manifolds. In his paper [Yau] from 1977, Yau proved not only Theorem 3.14 but also the following:

Theorem 4.2 (Yau Inequality). *Let X be a complex, projective, nonsingular variety of dimension s. Let $c_i := c_i(\mathcal{T}_X)$ be its i-th Chern class. If ω_X is ample, then*

$$(-1)^s \cdot c_1^s \leq (-1)^s \cdot \frac{2(s+1)}{s} \cdot c_2 \cdot c_1^{s-2}.$$

Later in 1987, Miyaoka followed up on [Miy1] with the paper [Miy2] and proved the following result:

Theorem 4.3 (Miyaoka Inequality). *Let X be a complex, projective, nonsingular variety of dimension s. Let $c_i := c_i(\mathcal{T}_X)$ be its i-th Chern class. If ω_X is ample, then*

$$c_1^2 \cdot D^{s-2} \leq 3 \cdot c_2 \cdot D^{s-2}.$$

for any numerically effective divisor D on X.

We note that both of them yield Theorem 3.14 in the case $s = 2$. In [CL], it is conjectured that these two inequalities are connected by a series of further inequalities. Our goal, naturally, is to apply these inequalities in Scenario 4.1. However, it is unclear why, or under what conditions, the canonical bundle of $\tilde{Y}$ is ample.

In general, the question of whether a variety has ample canonical bundle is fairly nontrivial. However, there is hope for our case: A few years after Hirzebruch had studied CBCs in the two-dimensional case, his student Bruce Hunt investigated them closely in dimension three. In section 2.3 of his PhD thesis [Hun], he verifies that for $s = 3$ and under certain conditions on the arrangement, the canonical bundle of $\tilde{Y}$ is ample. If this result generalizes to arbitrary s, one could apply Theorems 4.2 and 4.3 to obtain relations between the combinatorial data of the arrangement H. If these relations verify the condition (1.1) or a similar constraint, this would be a major breakthrough.

4.2 Prospects in Positive Characteristic

In the case $p := \mathrm{char}(\mathbb{k}) > 0$, a bound on $\mathrm{SG}_k(\mathbb{k}, m)$ can no longer be independent on m. For instance, if $\mathbb{k} = \mathbb{F}_p$ is the field with p elements, then $\mathbb{P}^s_{\mathbb{k}}$ is an SG_k-closed set itself, for any s and k. Saxena and Seshadhri gave a general bound in [SS], which states that

$$\mathrm{SG}_k(\mathbb{k}, m) \leq 9k \cdot \log(m) \tag{4.1}$$

for any field $\mathbb{k}$. However, there is no dependence on the characteristic p. For large enough p, we expect a bound that is closer to those in characteristic 0. We propose the following

Conjecture 4.4. *Let $\Bbbk$ be a field of positive characteristic $p > 0$. Then,*

$$\mathrm{SG}_k(\Bbbk, m) = O(k \cdot \log_p(m))$$

In order to approach this conjecture, we must first note that the Miyaoka-Yau Inequality (Theorem 3.14) fails to hold in general over fields of positive characteristic, as shown in [Eas]. However, similar relations in positive characteristic are also well-known, since 1991. We quote a theorem from [Mor]:

Theorem 4.5 (Moriwaki's Inequality). *Assume that $p = \mathrm{char}(\Bbbk) > 0$. Let X be an s-dimensional nonsingular projective variety with an ample divisor D. Let $\mathscr{E}$ be a locally free sheaf of rank r on X which is p-semistable with respect to D. Assume $s \geq 2$. Then,*

$$\left(r-1\right) \cdot \left(c_1(\mathscr{E})^2 \cdot D^{s-2}\right) \leq 2r \cdot \left(c_2(\mathscr{E}) \cdot D^{s-2}\right) \tag{4.2}$$

provided that $r \leq 3$ or $s = 2$.

While this inequality does not depend on p, [SB] provides a result for surfaces that does:

Theorem 4.6. *Let X be a surface over a field $\Bbbk$ of characteristic $p > 0$. Assume that Ω_X is K_X-stable. Then,*

$$c_1^2(\mathscr{E}) - 4c_2(\mathscr{E}) \leq K_X^2 \Big/ 4p^2$$

for every locally free sheaf $\mathscr{E}$ on X. In particular,

$$c_1^2 \leq \frac{16p^2}{4p^2 - 1} \cdot c_2, \tag{4.3}$$

where $c_i := c_i(\mathcal{T}_X)$ denotes the i-th Chern class.

We note that Theorem 4.5 for surfaces is just the Miyaoka-Yau Inequality (Theorem 3.14) with 4 instead of 3 as a constant. Furthermore, the results of Chapter 2 work over any algebraically closed field $\Bbbk$ and the condition that $p = \mathrm{char}(\Bbbk)$ may not divide n is neglible, because we are mainly interested in large values of p and small values of n. Finally, we note that the Gauss-Bonnet-Formula (Theorem 3.18) can be translated to positive characteristic – one then needs to define the Euler characteristic via ℓ-adic or étale cohomology and the proof has to be adjusted. Results like Proposition 2.22 and Corollary 2.24 also carry over to this case, as long as we are dealing with projective varieties. For $k = 2$, we could therefore attempt to generalize Kelly's proof to fields of finite characteristic. In this case, Kelly's Trick (Proposition 1.14) suggests the following conjecture, which would be further evidence for the validity of Conjecture 4.4.

Conjecture 4.7. *Let $\Bbbk$ be a field of characteristic $p > 0$. Assume that $X \subseteq \mathbb{P}^s_{\Bbbk}$ is an* $\mathrm{SG_2C}$ *with $|X| < 3p$. Then, X is contained in a projective plane.*

The major problem is that the constant 4 of Theorem 4.5 is simply too large. Let us assume that $\mu \in \mathbb{Q}$ is such that

$$c_1^2 \le \mu \cdot c_2 \tag{4.4}$$

holds for the variety $\tilde{Y}$, now constructed over a field of finite characteristic, but otherwise exactly as in Scenario 3.1. In particular, $t_2 = 0$. Since the formulas from Propositions 3.6 and 3.13 remain valid, we can caluclate

$$\begin{aligned}
n^{2-\ell} \cdot (\mu c_2 - c_1^2) = & \quad n^2\mu(3 - 2m + f_1 - f_0) \\
& + n^2(-9 - 3f_1 + 4f_0 + 5m) \\
& + 2n\mu(m - f_1 + f_0) + 2n(-2m + f_1 - f_0) \\
& + \mu f_1 + (-f_1 + f_0 - m) \\
= & \quad n^2(3\mu - 9 + f_1(\mu - 3) + f_0(4 - \mu) + (5 - 2\mu)m) \\
& + 2n(m(\mu - 2) + f_1(2 - \mu) + f_0(\mu - 2)) \\
& + f_1(\mu - 1) + f_0 - m \\
= & \quad f_1 \cdot (n^2\mu - 3n^2 + 4n - 2n\mu + \mu - 1) \\
& + f_0 \cdot (4n^2 - n^2\mu + 2n\mu - 4n + 1) \\
& + m \cdot (5n^2 - 2n^2\mu + 2n\mu - 4n - 1) \\
& + 3n^2\mu - 9n^2.
\end{aligned}$$

Since we can assume $m \ge 6$ and $\mu \ge 3$, the sum of the last two lines can easiely be seen to be negative. Thus,

$$\begin{aligned}
0 &< f_1 \cdot (n^2\mu - 3n^2 + 4n - 2n\mu + \mu - 1) + f_0 \cdot (4n^2 - n^2\mu + 2n\mu - 4n + 1) \\
&= \sum_{r \ge 2} \big(\underbrace{(n^2\mu - 3n^2 + 4n - 2n\mu + \mu - 1)}_{a} \cdot r + \underbrace{(4n^2 - n^2\mu + 2n\mu - 4n + 1)}_{b}\big) \cdot t_r
\end{aligned}$$

If we want to deduce $t_3 > \sum_{r \ge 4} -(ar + b)t_r \ge 0$ by arguing that $ra + b \ge 0$ for $r \ge 4$, we require $3a + b \ge 0$ and $4a + b \le 0$. This translates to

$$\begin{aligned}
(2n^2 - 4n + 3) \cdot \mu &\ge 5n^2 - 8n + 2 \\
(3n^2 - 6n + 4) \cdot \mu &\le 8n^2 - 12n + 3
\end{aligned} \tag{4.5}$$

Now, since $(n-3)^2 \ge 0$, we get $(8n^2 - 12n + 3) \le 3 \cdot (3n^2 - 6n + 4)$ with equality if and only if $n = 3$, so we know that $\mu \le 3$ follows from (4.5) and we can achieve equality only in the case $n = 3$ (see also Figure 4.1). Thus,

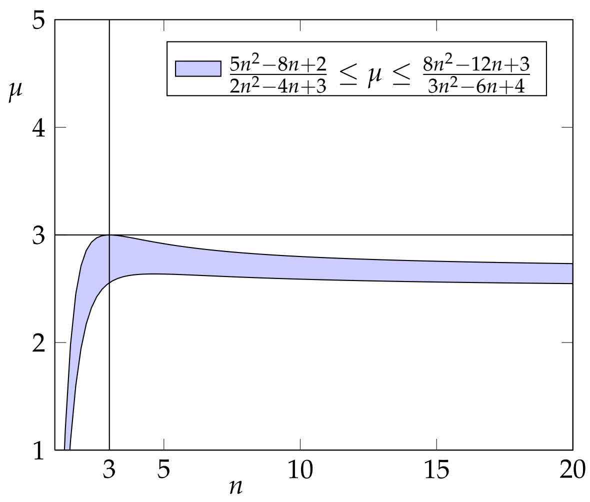

Figure 4.1: Visualization of (4.5).

the constant $\mu = 3$ and the choice of $n = 3$ in Theorem 3.21 are the only parameters that permit this proof strategy.

We already know that $\mu = 3$ does not hold in general for surfaces in finite characteristic. Thus, our question is:

Question 4.8. *Let $\Bbbk$ be a field of finite characteristic p and $Y = \mathbb{P}^2_{\Bbbk}(\sqrt[n]{h})$ for an ℓ-building h. Does Y satisfy the Miyaoka-Yau Inequality under the condition that $t_2(2, H) = 0$ for $H = \mathcal{Z}(h)$?*

If we could positively answer Question 4.8, then we could also prove Conjecture 4.7 by only slightly modifying Kelly's proof.

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List of Symbols

$\sqcap$	scheme-theoretic intersection (page 7).
$\dashrightarrow$	rational map between varieties (page 7).
$\varnothing$	the empty set.
$\sqrt[n]{x}$	set of n-th roots of x (page 34).
$\mathbb{A}^s$	affine s-space over the field $\Bbbk$ (page 8).
$A(X)$	Chow ring $A(X) = \bigoplus_k A^k(X)$ of X (page 22).
$A^k(X)$	group of k-cycle classes on X (page 22).
A_f	localization of the commutative ring A by $f \in A$ (page 6).
$A[IT]$	blow-up algebra of A in I (page 16).
A_P	localization of the commutative ring A by the multiplicatively closed set $A \setminus P$ (page 6).
Bl	blow-up construction (page 16, 18).
$\mathcal{B}_\pi$	branch locus of a morphism (page 30).
$\beta^{\mathsf{T}}(Z)$	strict transform of Z under the blow-up map β (page 20).
$\chi(X)$	topological Euler characteristic of a complex variety (page 38).
$\mathbb{C}$	the field of complex numbers.
$\mathrm{ch}(\mathscr{E})$	exponential Chern character of $\mathscr{E}$ (page 27).
$\mathrm{td}(\mathscr{E})$	Todd class of $\mathscr{E}$ (page 27).
$\mathrm{char}(\Bbbk)$	characteristic of $\Bbbk$.
$c_k(\mathscr{E})$	the k-th Chern class of $\mathscr{E}$ (page 25).
$\mathrm{codim}_X(Z)$	codimension of Z in X, i.e. $\dim(X) - \dim(Z)$.
$c_T(\mathscr{E})$	the Chern polynomial of $\mathscr{E}$ (page 25).
δ_{ij}	the Kronecker delta (page 48).

$D(f)$	open subset of an affine variety where the function f does not vanish (page 6).
$\deg(\pi)$	degree of a finite morphism (page 29).
$\deg(Z)$	degree of a projective variety (page 7).
$\operatorname{div}(f)$	rational cycle of zeros and poles of f (page 22).
$D_*(f)$	open subset of a projective variety where the homogeneous element f does not vanish (page 7).
$\mathscr{E}^\vee$	dual of a sheaf of $\mathcal{O}$-modules, i.e. $\mathscr{E}^\vee = \mathscr{H}om_{\mathcal{O}}(\mathscr{E}, \mathcal{O})$.
e_i	the i-th formal generator of a direct sum (page 46).
e_π	ramification index (page 31).
$\varphi^\sharp$	sheaf component of a morphism of schemes (page 6).
$\varphi^\star(\mathcal{I})$	inverse image ideal sheaf of $\mathcal{I}$ under φ (page 18).
ϕ_h	morphism to projective space, associated to a globally generated line bundle h (page 47).
$\mathbb{F}_p$	the field with p elements (page 64).
f_π	inertia degree (page 32).
$\operatorname{Frac}(R)$	Field of fractions of a domain R.
$\Gamma(\varphi)$	graph of a rational map φ (page 16).
H_λ	scheme-theoretic intersection of the irreducible components of an arrangement, indexed by λ (page 33).
$\mathscr{H}om_{\mathcal{O}}(\mathscr{E}, \mathscr{F})$	the hom-sheaf of two $\mathcal{O}$-modules $\mathscr{E}$ and $\mathscr{F}$ (page 40).
$\mathcal{H}^q(X, \mathscr{E})$	q-th sheaf cohomology group of the sheaf $\mathscr{E}$ (page 38).
$H^q(X, R)$	q-th singular cohomology group with coefficients in a commutative ring R (page 38).
$H_q(X, R)$	q-th singular homology group with coefficients in a commutative ring R (page 38).
$H(r)$	the set of r-points of an arrangement H (page 39).
$\int_X \alpha$	degree of a rational equivalence class $\alpha \in A(X)$ (page 25).
$\sqrt{I}$	radical of the ideal I (page 6).
$\operatorname{im}(\varphi)$	image of the morphism φ.
$\ker(\varphi)$	kernel of the morphism φ.
$I_*(Z)$	homogeneous ideal corresponding to the closed subset Z of a projective scheme (page 7).

$\mathcal{I}(Z)$	ideal sheaf corresponding to a closed subscheme Z (page 7).
$I(Z)$	ideal corresponding to a closed subset Z (page 6).
$\Bbbk$	a field (usually algebraically closed) (page 7).
$\Bbbk(X)$	function field of the variety X (page 6).
$K^{\times n}$	subgroup of $K^\times$ of all n-th powers (page 50).
K_X	canonical divisor on X (page 40).
$\mathrm{kod}(X)$	Kodaira dimension of a projective variety X (page 56).
$\mathscr{L}$	usually a line bundle.
λ	usually a multiindex $\lambda \subset \{0, \ldots, \ell\}$ (page 33).
$L + L'$	linear span of two linear varieties (page 8).
$L^\perp$	geometric dual of a linear projective variety (page 9).
$\mathrm{len}_A(M)$	length of the A-module M (page 22).
$[L : K]$	degree of a field extension $K \subseteq L$.
$\lambda_H(P)$, $\lambda(P)$	indices of the components of an arrangement H meeting at the point P (page 33).
$\mathfrak{m}_P$	maximal ideal of the local ring at P (page 6).
$M^\sim$	quasi-coherent sheaf associated to a module M (page 6).
$\mathbb{N}$	the natural numbers.
$\mathcal{O}_{X,P}$	local ring at P (page 6).
Ω_X	sheaf of relative differentials of X (page 40).
ω_X	canonical sheaf of X (page 40).
$\mathbb{P}^s$	projective s-space over the field $\Bbbk$ (page 7).
$\mathrm{Proj}(S)$	set of homogeneous prime ideals of a graded ring S, equipped with the standard scheme structure (page 7).
$\mathbb{P}\mathbf{roj}$	relative proj-construction (page 18).
$\mathbb{Q}$	the field of rational numbers.
$\mathbb{R}$	the field of real numbers.
$R^\times$	the group of units of a commutative ring R.
rank	rank of a matrix, or the rank of a free module over some ring.
$\mathrm{Rd}(H)$	the redundant part of an arrangement H (page 33).
$\mathcal{R}_\pi$	ramification locus of a morphism (page 30).