Demystifying the border of depth-3 circuits

Joint works with Pranjal Dutta & Prateek Dwivedi. [CCC'21, FOCS'21, FOCS'22]

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Schloss Dagstuhl, Leibniz-Zentrum

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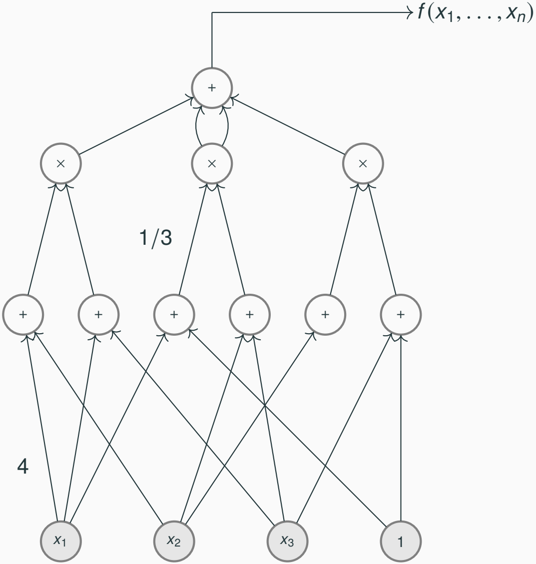
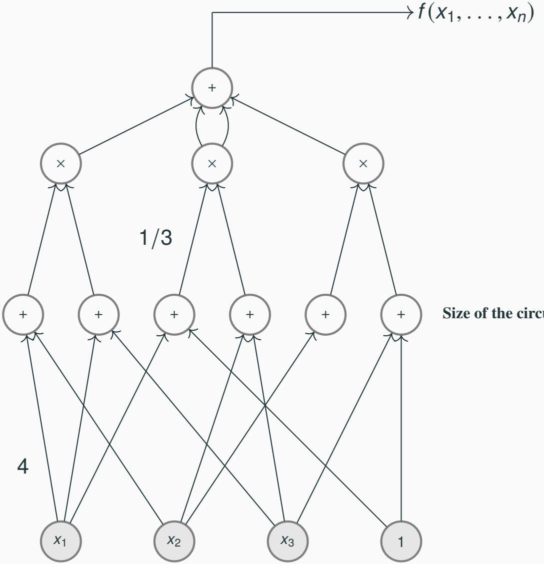
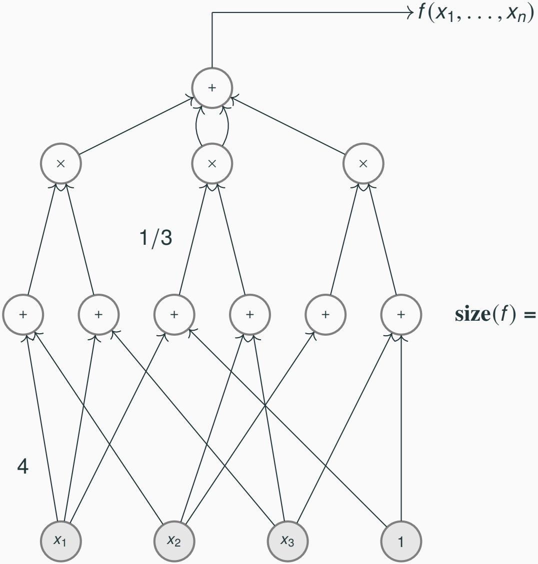




Size of the circuit = number of nodes + edges

Algebraic circuits– VP



size(f) = min size of the circuit computing f

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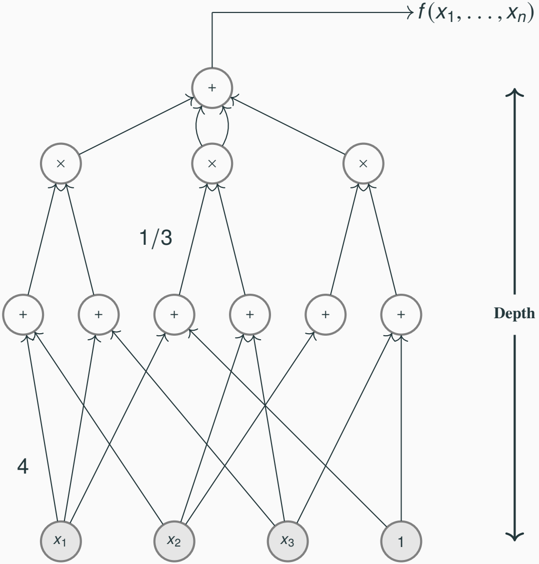

The determinant polynomial– VBP

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$$f_n \; := \; \det(X_n) \; = \; \sum_{\pi \in S_n} \operatorname{sgn}(\pi) \prod_{i=1}^{n} x_{i,\pi(i)} \; .$$

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- VBP: The class VBP is defined as the set of all sequences of polynomials $(f_n)_n$ with polynomially bounded $\mathrm{dc}(f_n)$.

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Valiant's Conjecture [Valiant 1979]

VBP $\neq$ VNP & VP $\neq$ VNP. Equivalently, $\mathrm{dc}(\mathrm{perm}_n)$ and $\mathrm{size}(\mathrm{perm}_n)$ are both $n^{\omega(1)}$.

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Border Complexity and GCT

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The smallest r such that a *homogeneous* degree d polynomial h can be written as a sum of d-th power of linear forms ℓ_i, i.e. $h = \sum_{i=1}^{r} \ell_i^d$.

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- Recall: $h = \sum_{e_1,..,e_n} a_{e_1,...,e_n} x_1^{e_1} \cdots x_n^{e_n}$, is called **homogeneous** degree d polynomial if $\sum e_i = d$, for every tuple $(e_1, \ldots, e_n)$ with $a_{e_1,...,e_n} \neq 0$.

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- For any homogeneous polynomial h, $\mathsf{WR}(h)$ is *finite*.
- Often $\mathsf{WR}(h) \leq r$ is denoted as $h \in \Sigma^{[r]} \wedge \Sigma$ (homogeneous *depth-3 diagonal* circuits).

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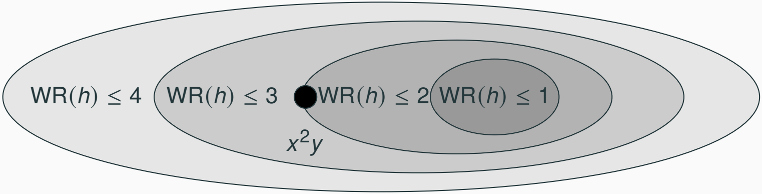

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The border Waring rank $\overline{\mathsf{WR}}(h)$, of a d-form h is defined as the smallest s such that $h = \lim_{\epsilon \to 0} \sum_{i \in [s]} \ell_i^d$, where $\ell_i \in \mathbb{F}(\epsilon)[\mathbf{x}]$, are homogeneous linear forms.

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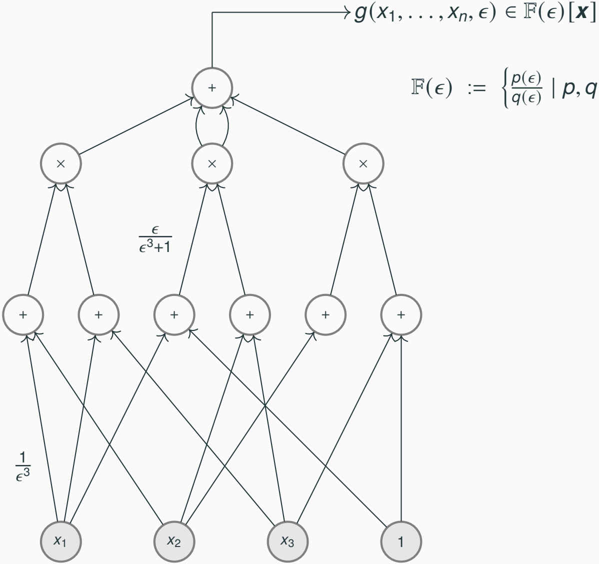
- We will work with *'approximative circuits'*.



$$\mathbb{F}(\epsilon) := \left\{ \frac{p(\epsilon)}{q(\epsilon)} \mid p, q \in \mathbb{F}[\epsilon], q(\epsilon) \neq 0 \right\}$$

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- Summary: g_0 is really something **non-trivial** and being 'approximated' by the circuit since $\lim_{\epsilon \to 0} g(\boldsymbol{x}, \epsilon) = g_0$.

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Algebraic Approximation [Bürgisser 2004]

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- $\overline{\text{size}}(h) \leq \text{size}(h)$. $[h = h + \epsilon \cdot 0.]$
- If g has circuit of size s over $\mathbb{F}(\epsilon)$, then one can assume that the highest degree of ϵ in g can be *exponentially large* 2^{s^2} [Bürgisser 2004, 2020].

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Algebraic approximation— $\overline{\text{VP}}$

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  - What about border depth-3 circuits (both upper bound and lower bound)?

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Border depth-3 fan-in 2 circuits are 'universal' [Kumar 2020]

Let P be *any* n-variate degree d polynomial. Then, $P \in \overline{\Sigma^{[2]}\Pi\Sigma}$, where the multiplication gate is $\mathsf{exp}(n, d)$.

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□

Proving Upper Bounds

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Remark. The result holds if one replaces the top-fanin-2 by arbitrary constant k.

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- Let $\Phi(T_i) =: \epsilon^{a_i} \cdot \tilde{T}_i$, for $i \in [2]$, where $a_i := \mathsf{val}_\epsilon(\Phi(T_i))$.

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- Suffices to compute $g_1 \mod z^d$ and take the limit!

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- Eliminate division and integrate (interpolate) to get $\Phi(f)/t_2 = \mathsf{ABP} \implies \Phi(f) = \mathsf{ABP} \implies f = \mathsf{ABP}$.

Proving Lower Bounds

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  - Since, $\mathsf{IMM}_{n,d} \in \mathsf{VBP}$, $\overline{\Sigma^{[k]}\Pi\Sigma} \neq \mathsf{VBP}$.

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- What does work (if at all!)?

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- Classical is about *impossibility* while in border, it is about *optimality*.

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- Recall the non-border lower bound proof, of making an ideal $\mathcal{I}_k = \langle \ell_1, \ldots, \ell_k \rangle$, such that $\mathsf{det}_n \neq 0 \bmod \mathcal{I}_k$, but $\Sigma^{[k]}\Pi\Sigma = 0 \bmod \mathcal{I}_k$.

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- The worst case:

$$f + \epsilon S \;=\; T_1 + T_2\,,$$

where T_i has each linear factor of the form $1 + \epsilon\ell$!

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  - Case II (intermediate): T_1 has one homogeneous factor (say ℓ_1) and ϵ-free part of all factors in T_2 are non-homogeneous (in $\boldsymbol{x}$). Non-homogeneous example: $(1+\epsilon) + \epsilon x_1$.
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Thank you! Questions?