A Largish Sum-of-Squares Implies Circuit Hardness and Derandomization

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Table of contents

Introduction: Sum-of-squares (SOS)

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Trivially, $S_{\mathbb{F}}(f) \leq 2 \cdot (|f|_0 + 1)$, for *any* $f \in \mathbb{F}[\boldsymbol{x}]$.

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- Suppose $f = \sum_{i=1}^{s} c_i \cdot f_i^2$. Assume, $|f_i|_0 = t_i$.
- Note, $|f_i^2|_0 \leq t_i^2$, for each $i \in [s]$.
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  - Assume, $\mathbb{F} = \mathbb{C}$.

Think as *quadratic*-system solving.

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  - To be of any help in complexity theory, polynomials *need* to be explicit. We would work with several definitions of explicitness.
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- **Overall Goal (informally):** Show that solving Open Problem implies $\mathsf{VP} \neq \mathsf{VNP}$ (and PIT $\in \mathsf{SUBEXP}$).

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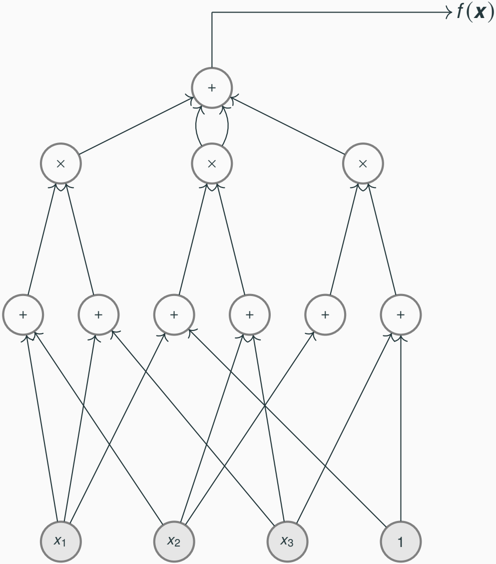
Basic Algebraic Complexity

Algebraic Circuits



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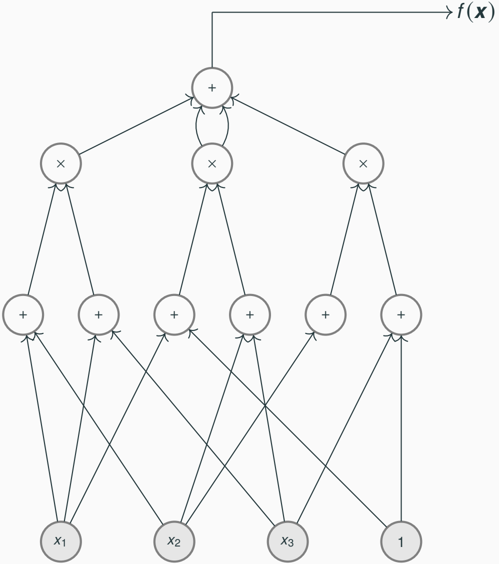

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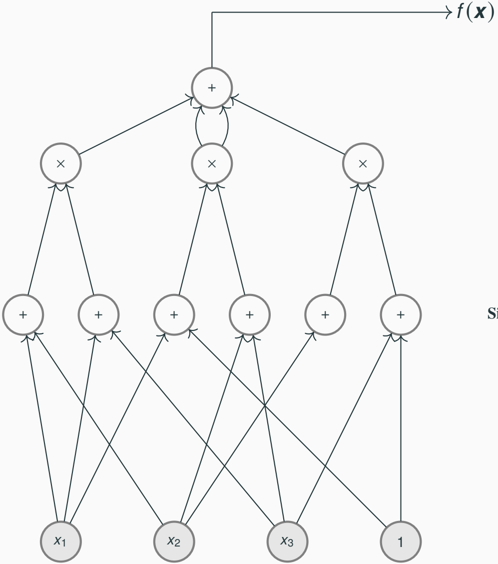

Size = number of nodes + edges

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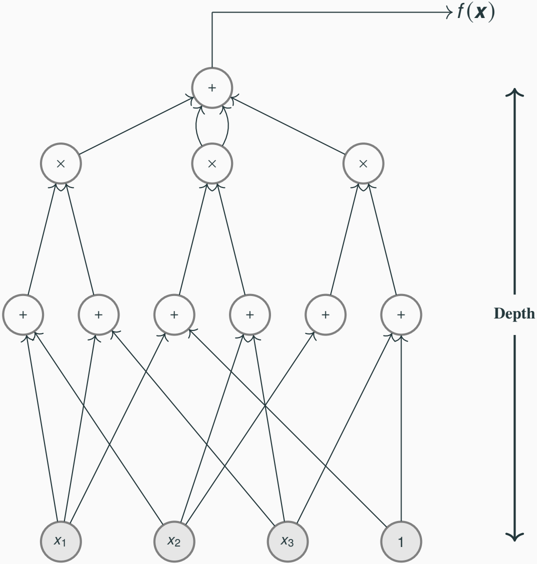

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- VNP is *exponentially* harder than VP $\implies$ PIT $\in$ QP.
- Efficient PIT $\overset{?}{\implies}$ VP $\neq$ VNP. *Explicitness* is important.

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Requires GRH to separate VP and VNP.

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- [Raz'08] Super-poly-elusive functions eluding degree-2 maps (generic *multivariate*).

SOS-hardness to $\mathsf{VP} \neq \mathsf{VNP}$

Theorem 1 (Dutta-Saxena-Thierauf'20)

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- Restrict the degrees of f_i to be $d \cdot o(\log d)$ and the top-fanin $s = d^{o(1)}$.
- A stronger SOS-hardness notion with *constant* ε, gives an *exponential* separation between VP and VNP. This proof has many technical differences.

SOS-hardness and VP vs. VNP

Main Lemma (SOS Decomposition)

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Let $\mathbb{F}$ be a field of characteristic $\neq 2$. Let $f(\boldsymbol{x})$ be an n-variate polynomial over $\mathbb{F}$ of degree d, computed by a circuit of size s. Then there exist $f_i \in \mathbb{F}[\boldsymbol{x}]$ and $c_i \in \mathbb{F}$ such that

$$f(\boldsymbol{x}) = \sum_{i=1}^{s'} c_i f_i(\boldsymbol{x})^2 ,$$

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Can we *improve* s' to $\mathrm{poly}(sd)$?

Algebraic branching programs (ABP). An ABP is a directed acyclic graph with a *starting vertex* s with in-degree zero, an *end vertex* t with out-degree zero. The edge labels are $a_1x_1 + \ldots + a_nx_n + c \in \mathbb{F}[\boldsymbol{x}]$, where $a_i, c \in \mathbb{F}$.

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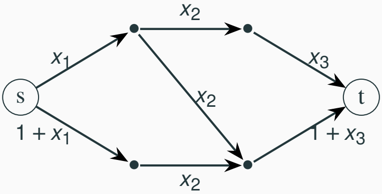

This ABP computes

$$x_1x_2x_3 + x_1x_2(1+x_3) + (1+x_1)x_2(1+x_3)$$

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- Write each product $f_i \cdot f'_i \ = \ 1/4 \cdot (f_i + f'_i)^2 \ - \ 1/4 \cdot (f_i - f'_i)^2$, which finally gives the desired decomposition.

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Recall Theorem 1: If an explicit $f_d(x)$ is SOS-hard i.e. $S_{\mathbb{F}}(f_d) \geq d^{1/2+\varepsilon}$ for $\varepsilon = \omega(\sqrt{\log\log d/\log d})$, then $\mathsf{VP} \neq \mathsf{VNP}$.

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- $P_{n,k}$ is very explicit and thus the family $\in \mathsf{VNP}$.

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- Thus, $S_{\mathbb{F}}(f_d) \leq d^{o(\varepsilon)+1/2+\varepsilon/2} = o(d^{1/2+\varepsilon})$, a contradiction!

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Sum-of-cubes (SOC) model and Blackbox-PIT

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  - Denote the *minimal size* by support-union $U_{\mathbb{F}}(f, s)$.

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- SOC is a *complete* model for $\mathrm{char}(\mathbb{F}) \neq 2, 3$ because for any $f(\boldsymbol{x})$:

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- A binomial counting argument shows that small size of $P_{n,k}$ implies $U_{\mathbb{F}}(f_d, d^{\varepsilon'}) = o(d)$, a contradiction!

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