How to factor objects?

Nitin Saxena
CSE@IITK

IMSc@60, Chennai
January 2023

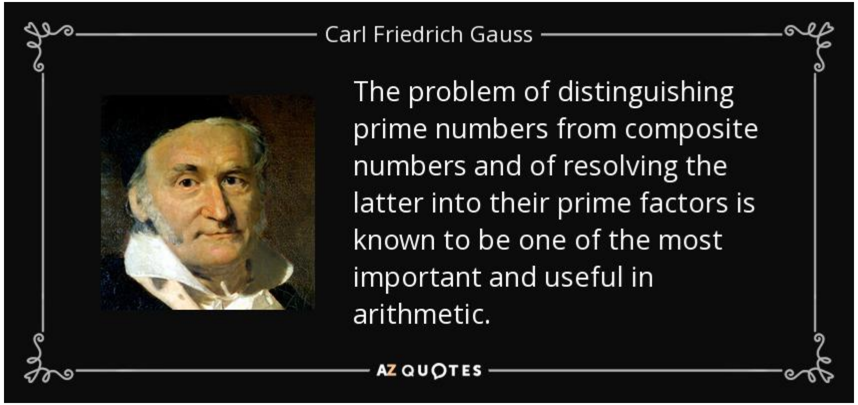
Base cases

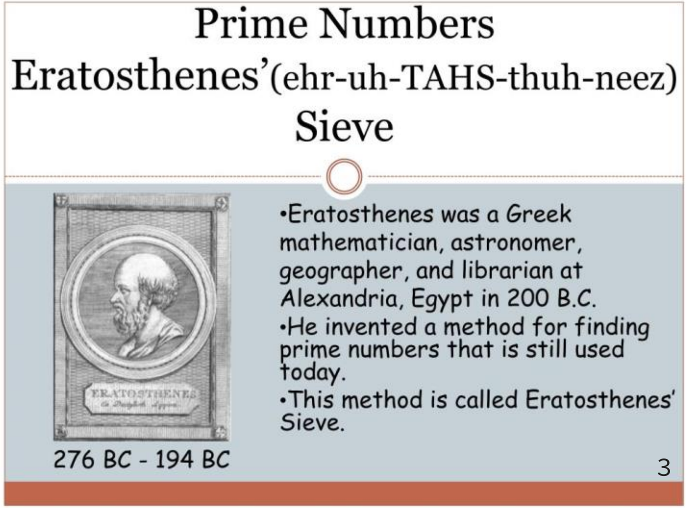
Integers

- Integer n factors *uniquely* into prime numbers.
  - Eg. $1092 = 2^2*3*7*13$

- Given n, can you factor it?
  - Input n in binary
  - 2^{(log n)^{0.3}} time not good enough
  - Number Field Sieve (1990s) factors via $x^2 = y^2 \bmod n$

- Hardness used in cryptosystems.
  - RSA, HTTPS, SSh, SFTP, Diffie-Hellman, ...



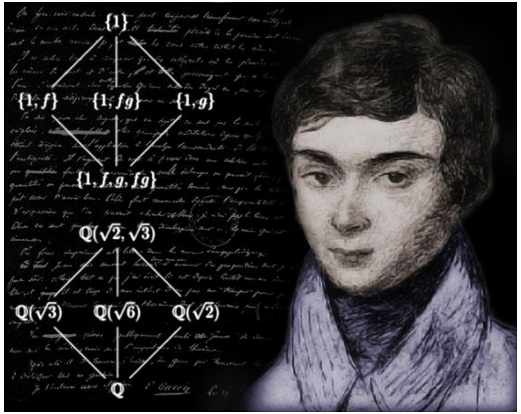


UNIVARIATE OVER INTEGERS

- Given polynomial $f(x) \in \mathbb{Z}[x]$, factor it.
  - $f = x^5 - x^4 - 4x^2 + x - 2$ factors
  - Roots have no formula
  - Irreducibility testing?

- [Lenstra,Lenstra,Lovász'82] solved this completely.
  - Factor mod $2, 2^2, 2^4, 2^8, \ldots$
  - Lift to integral factor using lattice theory
  - Useful in many *post-quantum* cryptosystems

Univariate over finite fields

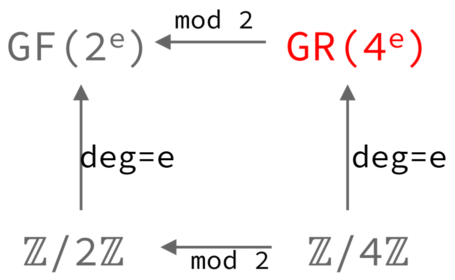


- Galois field GF(p^e) of size p^e and char = prime p.
- Given polynomial f(x) ∈ GF(p)[x], factor it.
  - f = x^2-2 factors mod 7
  - $\sqrt{2} = 3 \bmod 7$!
  - Irreducibility testing?

- [Berlekamp’67; Cantor-Zassenhaus’81] solved this practically.
  - Use Galois automorphism
  - Compute gcd of f(x) with x^p-x, $x^{p^2}-x$, $x^{p^3}-x$,...
  - Useful in crypto, coding theory, and computational algebra

Univariate over Galois rings

- Galois ring $GR(p^{ke})$ of size p^{ke} and characteristic = prime-power p^k.
- Given polynomial $f(x) \in GR(p^{ke})[x]$, factor it.
  - ➢ $f = x^2-2$ factors mod 7^2
  - ➢ $\sqrt{2} = 10 \bmod 7^2$!
  - ➢ Irreducibility testing?

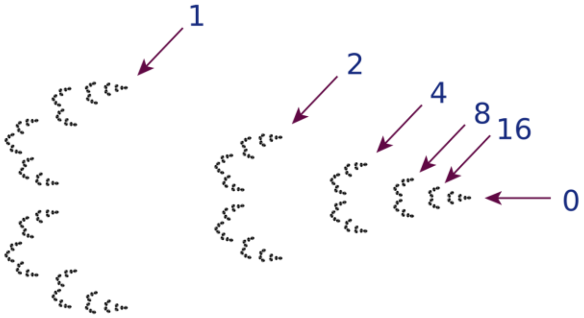


- This problem is OPEN.
- [Dwivedi,Mittal,S.'19] solved for k=4.
  - ➢ Factor $f(x)$ mod p, p^2, p^3, p^4.
  - ➢ Lifting from one to the next precision is *nontrivial*.
  - ➢ Eg. $f = x^2-p \bmod p^2$ vs $f = x^2-px \bmod p^2$

Univariate over p-adic numbers

- Hensel (1897) defined p-adic numbers Z_p.
  - $1+2p+3p^2+4p^3+5p^4+...$ converges to a number!
- Given polynomial $f(x) \in Z_p[x]$, factor it.
  - $f = x^2-2$ factors in 7-adic
  - $\sqrt{2} = 3 + 1*7 + 2*7^2 + 6*7^3 + ...$ in infinite digits!
  - Irreducibility testing?

- [Chistov'90; Cantor,Gordon'00] solved it efficiently.
  - Newton polytope of f(x),
  - coupled with p-adic metric,
  - reduces to mod p factoring.
  - Useful in computational number theory.

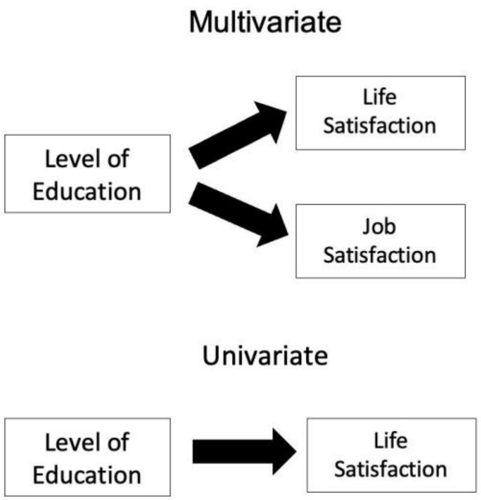


Multivariates

Multivariate sparse polynomials

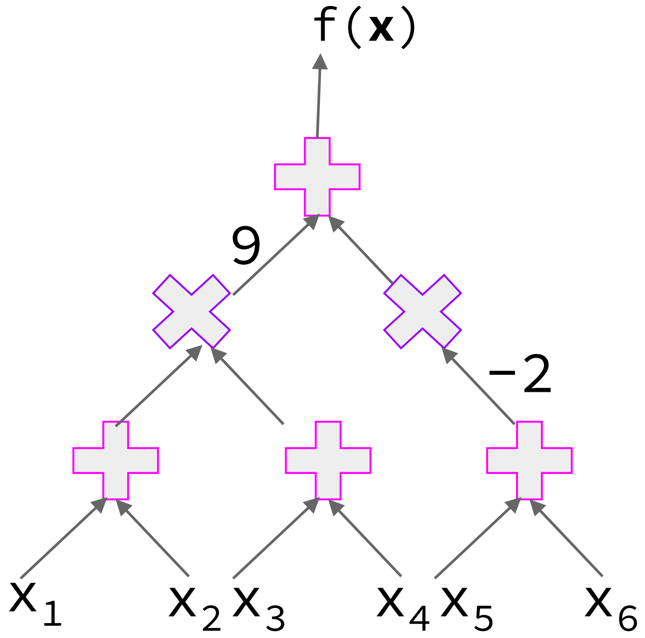
- Given polynomial $f(x_1,x_2,...,x_n) \in F[\mathbf{x}]$, factor it.
  - $f = (x_1^d-1)...(x_n^d-1)$ factors into
  - $g = (x_1^{d-1}+...+x_1+1)...(x_n^{d-1}+...+x_n+1)$.
  - Sparsity $s:=2^n$ blows-up to d^n.
  - => Factors can be very large!
- What if individual-degree d is constant?

- [Bhargava,Saraf,Volkovich'18] showed a quasipoly bound.
- [Bisht,S.'22] showed a poly bound for *symmetric* factors.
  - Newton polytope of $f(\mathbf{x})$
  - Relation between #vertices & #internal points.
  - Fast algorithm, by reducing to the base cases



In formula model

- Given polynomial $f(x_1,x_2,...,x_n) \in F[\mathbf{x}]$, factor it.
  - Input: is a formula of size s.
  - Output: is a formula of size =?



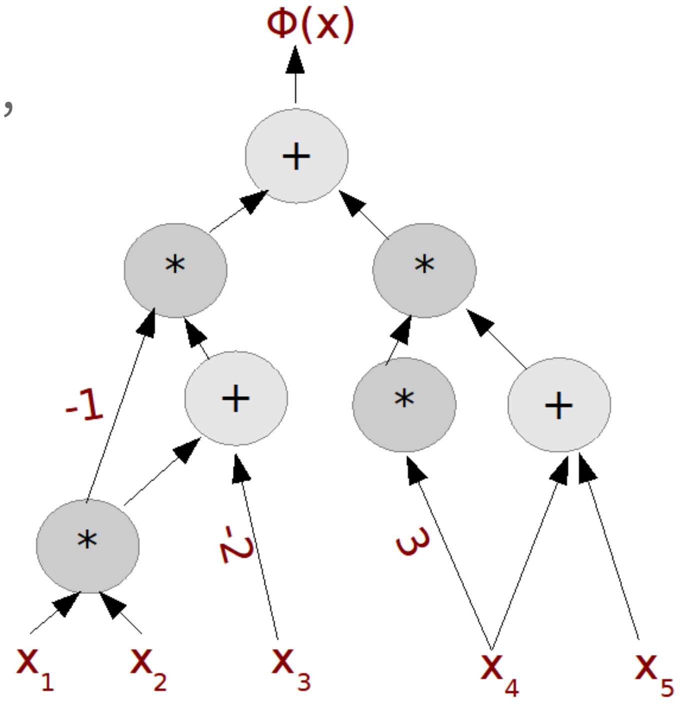
- Only quasipoly bound known.
- Poly bound is OPEN.
- **Open**: Could Newton iteration be done inside the model?

In circuit model

❖ Given polynomial $f(x_1,x_2,...,x_n) \in F[\mathbf{x}]$, factor it.
  ➢ Input: is a circuit of size s.
  ➢ Output: is a circuit of size =?

❖ [Kaltofen'87] showed a poly bound.
  ➢ degree not too `high'

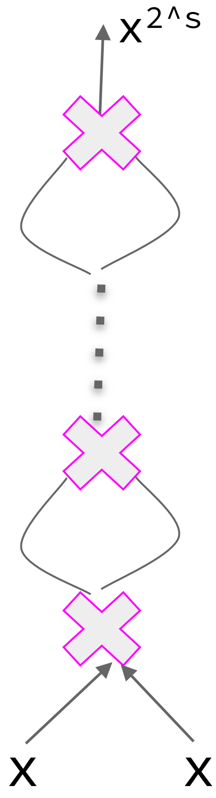
❖ **Corollary:** Newton iteration is doable inside circuits.



In a 'tougher' circuit model

- Given polynomial $f(x_1,x_2,...,x_n) \in F[\mathbf{x}]$, factor it.
  - Input: is a circuit of size s and degree 2^s.
  - Output: a factor of degree poly(s) of size =?

- It's an **open** question.
- [Dutta,S.,Sinhababu'18] showed a poly bound, when
  - degree of the radical of f is not too `high'.
- **Corollary**: all-roots-Newton-iteration is doable inside circuits.



Conclude with open problems

- **Question 1:** Fast integer factoring?
- Question 2: Fast polynomial factoring mod p^k?
- **Question 3:** General formula & circuit factoring?
- Question 4: Derandomization?

question

Thanks!

More details at:

cse.iitk.ac.in/users/nitin/