Algebra powers computation

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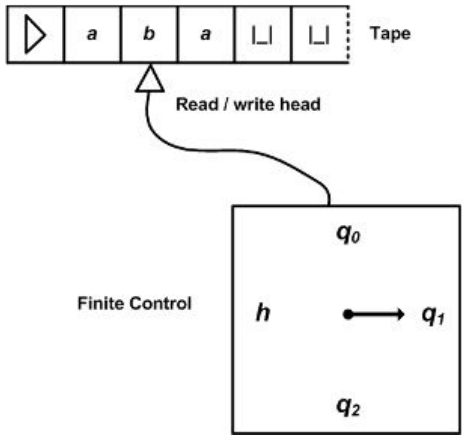
What's computing?

- Alan Turing (1936) postulated a simple, most general, mathematical model for computing – **Turing machine** (TM).

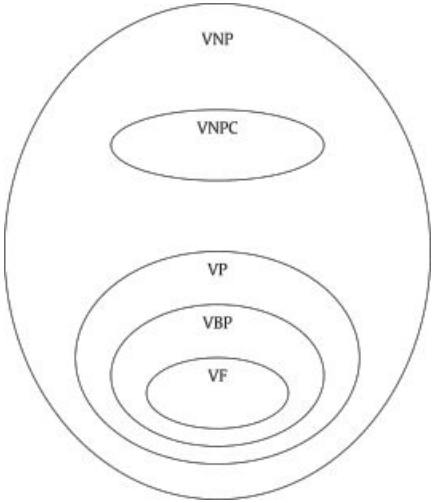
- **Algorithm** = TM is very much like a computer *program*.
  - TM is a real computer – highly iterative & trivial steps.

- How about an **electronic** circuit?
  - **Algebraically**, it's a neater model to capture real computation.

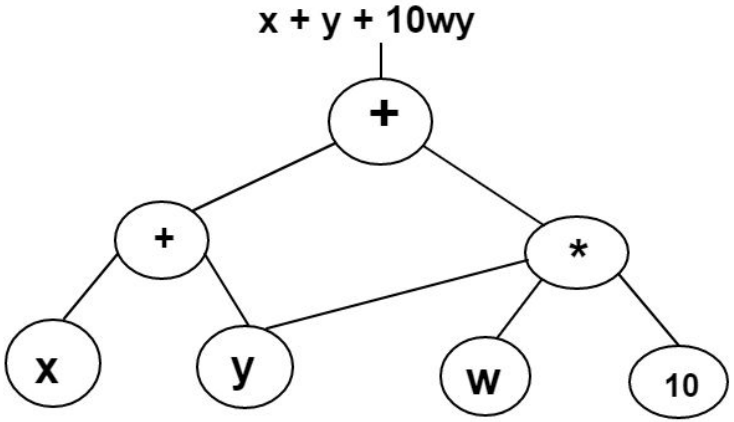


Turing (1912-1954)

Valiant: Algebraic circuits

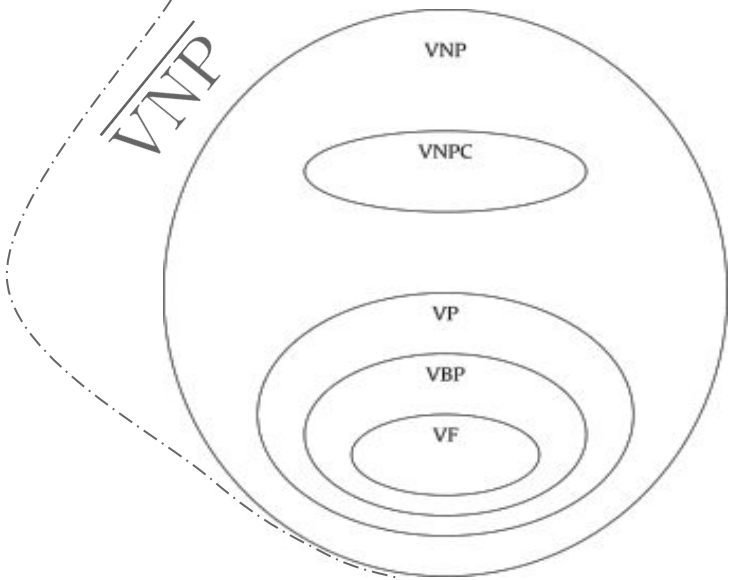


- Valiant (1977) formalized computation & resources using algebraic circuits.
  - Giving birth to his VP ≠ VNP question.
  - Or, the algebraic hardness question!
- Algebraic circuit has constants/variables, size, depth.



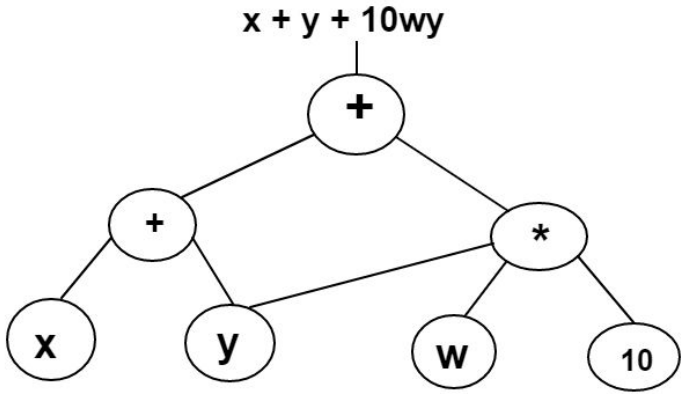
Leslie Valiant (1949-)

Valiant: Algebraic circuits



- **My work:** Study circuit problems and their properties.
  - Develop the relevant mathematics.

- **De-fictionalize** the above picture!
  - Progress has been impressive.
  - Withstands AI hype.



Zero or nonzero: PIT

$$\begin{aligned}(a_1^2+a_2^2+a_3^2+a_4^2)\,(b_1^2+b_2^2+b_3^2+b_4^2)\\ =(a_1b_1-a_2b_2-a_3b_3-a_4b_4)^2+(a_1b_2+a_2b_1+a_3b_4-a_4b_3)^2\\ +(a_1b_3-a_2b_4+a_3b_1+a_4b_2)^2+(a_1b_4+a_2b_3-a_3b_2+a_4b_1)^2.\end{aligned}$$

Euler's identity (1749)

- Question: Test whether a given circuit is ZERO.
  - ➢ Polynomial identity testing (PIT).

- OPEN Qn: Is PIT in deterministic polynomial time?

- Motivates new tools to study algebraic computation.

$$\begin{aligned}10 &= 1^2+1^2+2^2+2^2\\ 103 &= 2^2+3^2+3^2+9^2\\ 312 &= 2^2+4^2+6^2+16^2\end{aligned}$$

Lagrange's four-square theorem (1770)

$$(X+1)^n \equiv X^n+1 \bmod n$$
$$\iff n \text{ is } ?$$

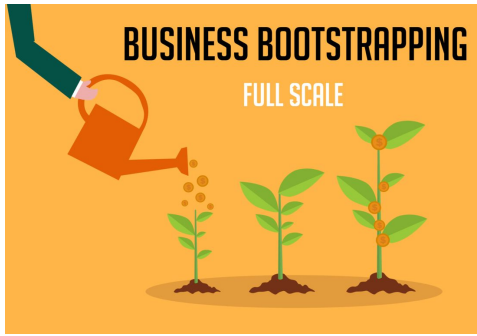
Zero or nonzero: PIT

$$(X+1)^n \equiv X^n + 1 \bmod n$$

- Primality testing.
- Blackbox algorithms/
  - Lower bounds/ Learning algos.
- Incidence-geometry in identities, over all fields.
  - Higher-dimension rank concepts.
- Duality in circuits.
  - Diagonal depth-3 or 4.

$x_1, x_2, x_1^2 + x_1 x_2$ are dependent.

$(x_1 + \cdots + x_n)^d$, as $f_1(x_1) \cdots f_n(x_n)$?



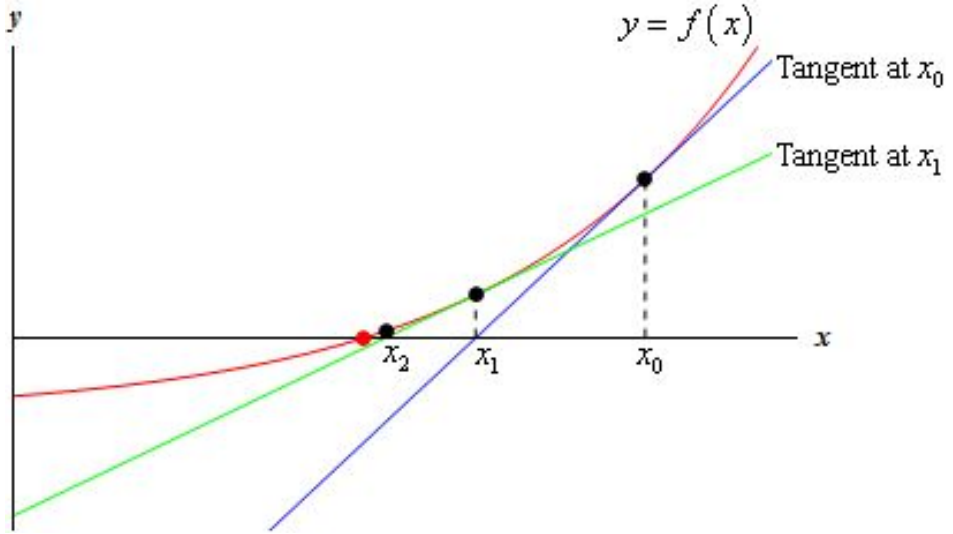
- Bootstrapping in circuits.
  - Tiny circuits
  - Sum-of-squares (SOS).

$$f \overset{?}{\neq} f_1(x)^2 + \cdots + f_n(x)^2 \implies ?$$

Algebraic Algorithms

Computational Algebra

- All-roots Newton iteration
  - Non-simple roots?
  - N-variate circuit version.



- GCD. Factoring polynomials.
  - Mod primes, prime-powers, p-adics
  - Circuit models
  - Approximative circuits

$$\gcd(x^2 - 2, x + 4) = x - 3$$

$$\sqrt{2} = 3 + 1 \times 7 + 2 \times 7^2 + 6 \times 7^3 + \cdots$$

$$x = 0 = x \cdot y - 1 \text{ has } \underline{\text{root}} = (\epsilon \to 0, 1/\epsilon \to \infty)$$

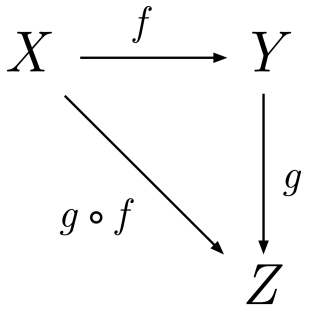
But, has no actual root!

Computational Algebra

- Algebraic dependence criteria
- Morphism problems in algebras, graphs
- Roots counting
- Compute Zeta function analogs
  - Infinite information?

$\mathbb{Q}$–dependence of e and π?

$x_1 + 1, x_1 + x_2, x_1^2 + x_1 x_2$ are dependent?



$x^2 = 0 \bmod p^2$ has p roots

$F(x, y) = 0$

$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = 0$

with $s \in \mathbb{C}$

Hardest question on earth since 1859.

$P(t) := \sum_{i \geq 0} N_{p^i}(F)/p^{2i} \cdot t^i \stackrel{?}{=} 1/(1-t)$?

Is it always this simple?

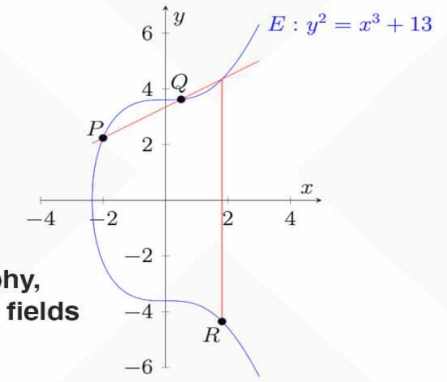
Engineering

Feeling insecure?

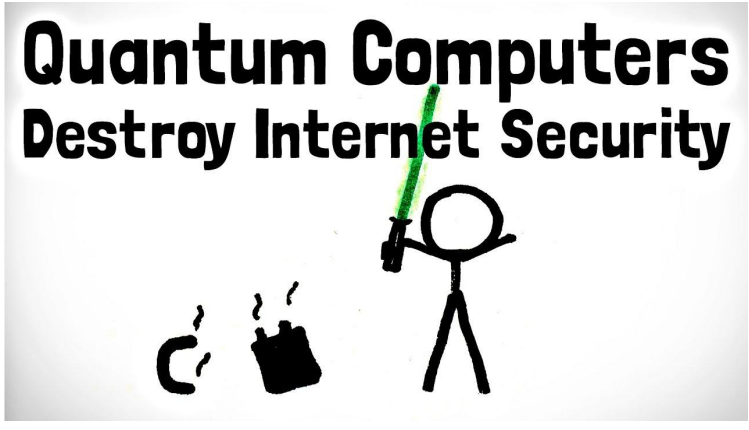
axiros
Lasting Advantage

Elliptic Curve Cryptography

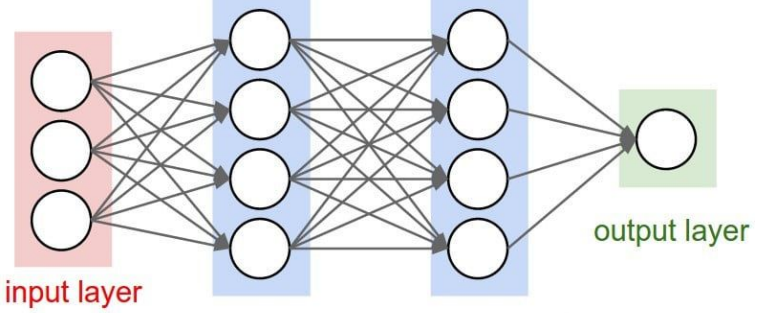
A choice for public-key-cryptography, based on elliptic curves over finite fields

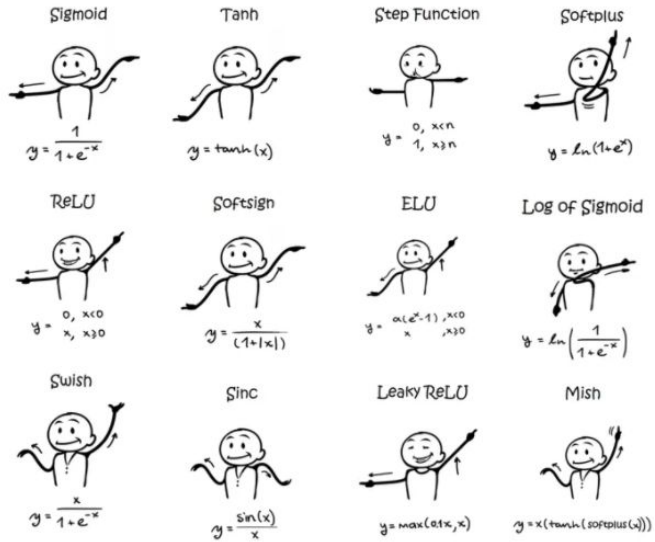


- Cryptography builds on algebra.
  - Number theory
  - Curves, counting, morphism
  - Multivariate systems
- Post-quantum world requires new protocols.
  - Avoid *abelian* groups.
  - Use complicated geometries,
  - morphisms,
  - and lattices.
- Inspiration from NP-hard problems?
  - interesting beyond quantum hype

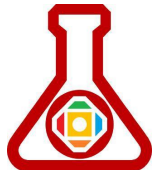


AI: Statistical patterns?





- AI/Machine Learning: Decision-making by Circuits, using *Statistics POV*!
  - Artificial Neural Networks (ANN).
- ANN is a specialized algebraic circuit.
  - Activation functions are real algebraic.
- A Center @IITK to solve practical, at-scale, problems using AI methods.
  - Visit (Center for Developing Intelligent Systems) www.iitk.ac.in/cdis/ www.cse.iitk.ac.in/users/nitin/



THANKS!