Demystifying the border of depth-3

Joint works with Pranjal Dutta & Prateek Dwivedi. [CCC'21, FOCS'21, FOCS'22]

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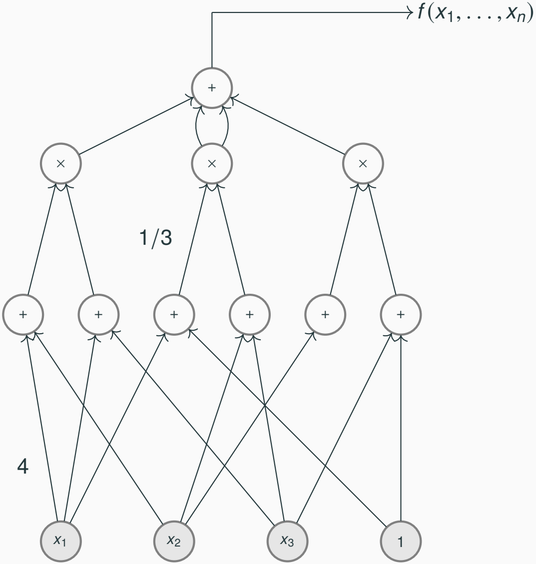
Algebraic circuits– VP



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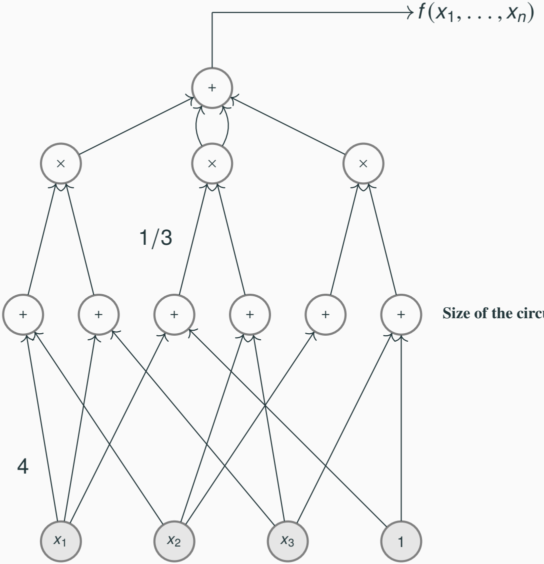
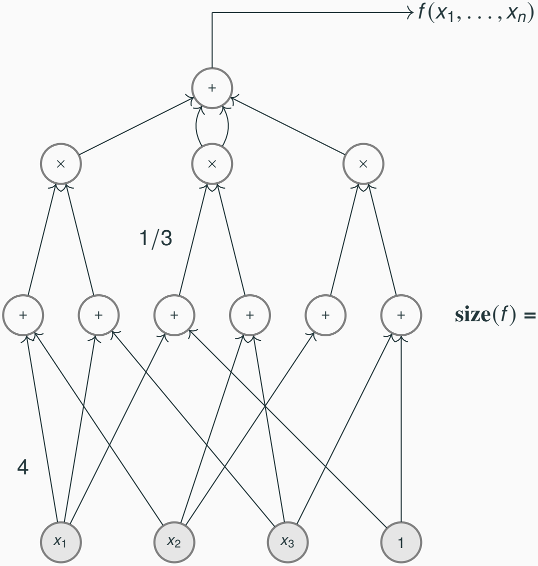


Size of the circuit = number of nodes + edges

Algebraic circuits– VP



size(f) = min size of the circuit computing f

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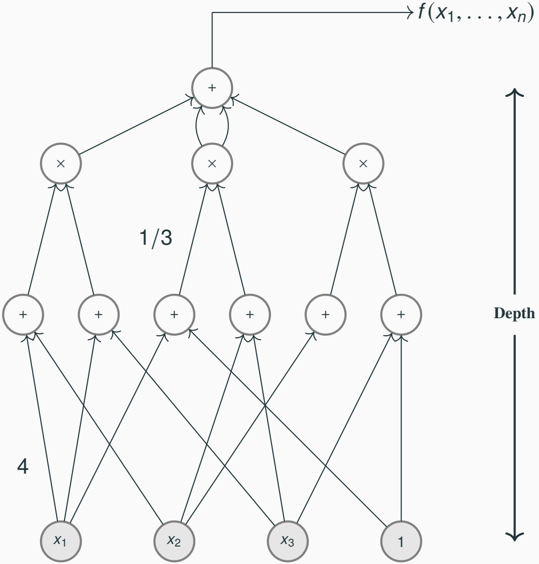

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- Relates tightly to Algebraic Branching Programs ABP, or IMM: Iterated Matrix Multiplication.

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- The minimum dimension of the matrix X_s to compute f, is called the **permanental complexity** $\text{pc}(f)$.

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Valiant's Conjecture [Valiant 1979]

$\mathsf{VBP} \neq \mathsf{VNP}$ & $\mathsf{VP} \neq \mathsf{VNP}$.

Equivalently, $\mathsf{dc}(\mathsf{perm}_n)$ and size(perm_n) are both $n^{\omega(1)}$.

Border Complexity and GCT

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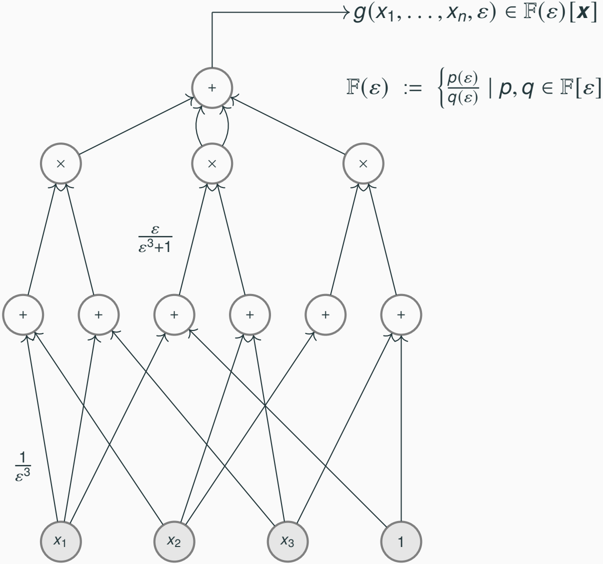
- This motivates a new model: *'approximative circuit'*.



$\mathbb{F}(\varepsilon) := \left\{ \frac{p(\varepsilon)}{q(\varepsilon)} \mid p, q \in \mathbb{F}[\varepsilon], q(\varepsilon) \neq 0 \right\}$ is the fn.field

Algebraic approximation

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- Summary: g_0 is **non-trivially** 'approximated' by the circuit, since $\lim_{\varepsilon \to 0} g(\boldsymbol{x}, \varepsilon) = g_0$.

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- $\overline{\text{size}}(h) \le \text{size}(h) \le \exp(\overline{\text{size}}(h))$.
- Curious eg.: Complexity of degree-s factor of a size-s polynomial? VP? VNP?

Border Depth-3 Circuits

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- How about $\overline{\Sigma^{[2]}\Pi\Sigma}$?

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3. Replace x_i by $\varepsilon \cdot x_i$ to get that

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Proof of Kumar's result

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Proving Upper Bounds

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Remark. The result holds if one replaces the top-fanin-2 by arbitrary constant k.

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- $\lim_{\varepsilon \to 0} g_1 = \lim_{\varepsilon \to 0} \partial_z \Phi(T_1/T_2) = \lim_{\varepsilon \to 0} \partial_z \Phi(f/T_2)$.

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- Both $\Phi(T_1)$ and $\Phi(T_2)$ have $\Pi\Sigma$ circuits (they have z and ε).

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- Note: *Definite* integration requires setting $z = 0$ in $\Phi(T_1/T_2) + 1$; that's why we need power-series in z.

Proving Lower Bounds

Looking for finer lower bounds

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Thank you! Questions?