A non-Turing model of computation

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(**Thanks to the artists for their images.*)

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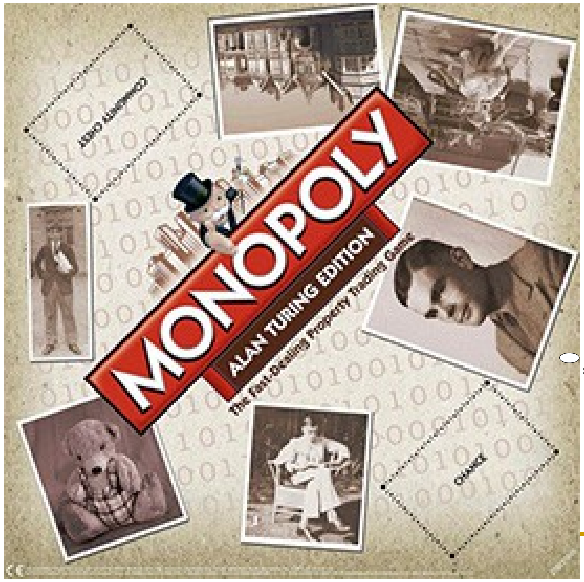
Contents

- Church & Turing
- Valiant: Algebraic circuits
- Warmup: Determinant
- Valiant's question
- Zero or nonzero
- Applications

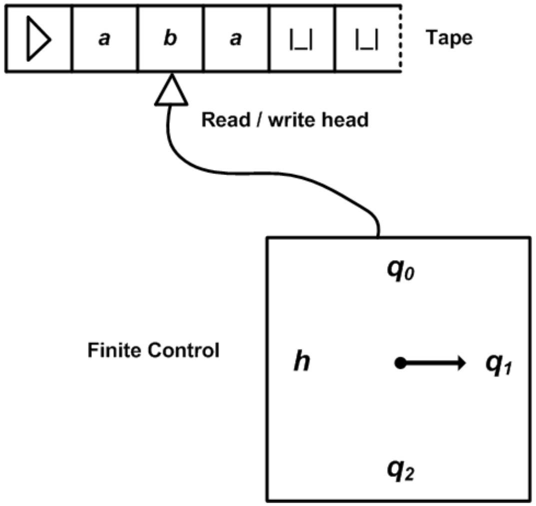
What's computing?

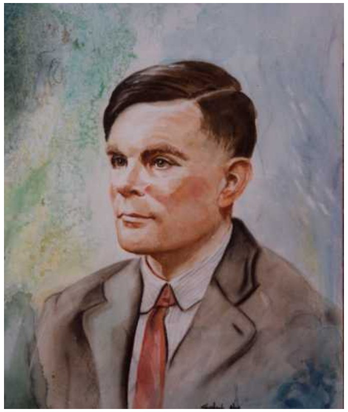
Church (1903-1995)

- The first response was by Alonzo Church (1935-6).
  - Using effective computability based on his λ-calculus.
- Soon, Alan Turing (1936) postulated a simple, most general, mathematical model for computing – Turing machine.



100th bday!
2012.





Turing (1912-1954)

Church & Turing

Hilbert (1862-1943)

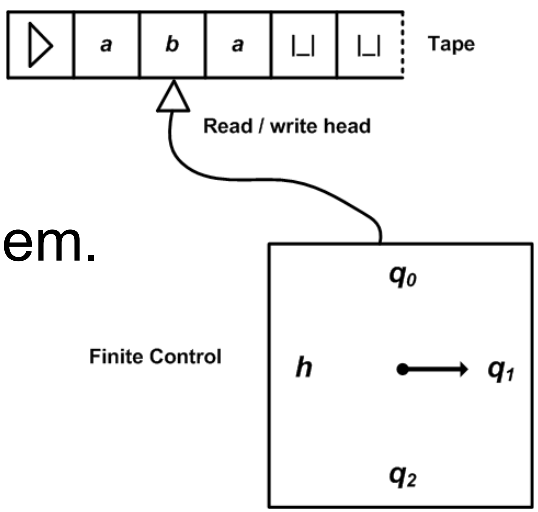
- Computation is: whatever can be simulated on a Turing machine (TM).
- TM invented to resolve Hilbert (1928)'s question--

  "***design an <u>algorithm</u> to decide whether a given statement is provable from the axioms using logic*".**

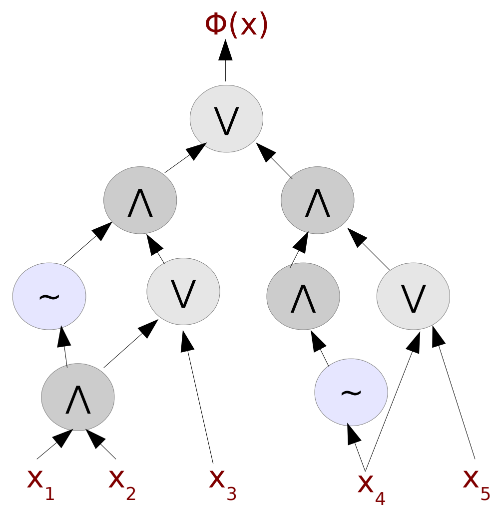
  Entscheidungsproblem

- The answer requires defining 'algorithm'.
  - hence, 'computation' requires a new mathematical framework.
- Algorithm is very much like a *program*.
  - TM is like a *real computer*-- highly iterative & trivial steps.

Church & Turing



- TM defines time & space complexity of a problem.
  - Asymptotics: as a function of input size n.
- OPEN Qn: Is there a problem that *requires* 2^n time on a TM?
  - Consider SATisfiability problem. Boolean formulas.
  - You get the famous P≠NP question.



- It's unclear how to prove the impossibility of a *faster* algorithm.
  - Since math proofs need an algebraic or geometric structure.
- So, let us look at an algebraic analogue---

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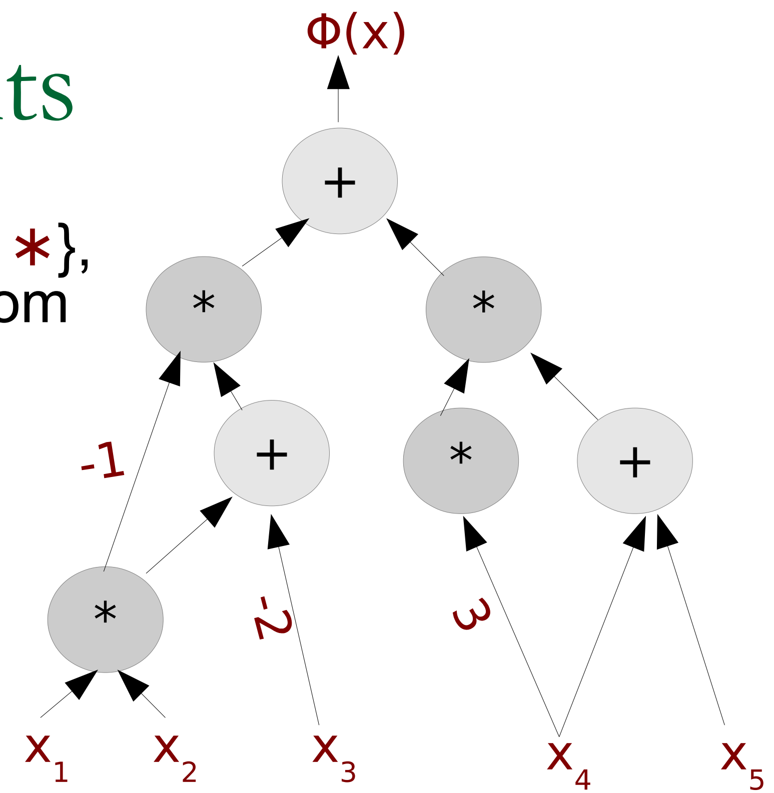
Valiant: Algebraic circuits



- An arithmetic circuit Φ has gates {+, ∗}, variables $\{x_1,\ldots,x_n\}$ and constants from some field F.
  - Defines size, depth, fanin, fanout.
- An arithmetic circuit is an *algebraically neat model* to capture real computation.
  - It's complete for polynomials!
- Valiant (1977) formalized computation & resources using algebraic circuits.
  - Giving birth to his VP≠VNP question!

Leslie Valiant (1949-)

Valiant: Algebraic circuits

- In n-variate n-degree polynomial $f(\mathbf{x})$, there are at most ${}^{n+n}C_n \approx 2^n$ monomials.
  - "Usually", that's the circuit-*size* complexity of $f(\mathbf{x})$.
  - What's circuit-*depth* of $f(\mathbf{x})$?
- OPEN Qn: Is there explicit polynomial $f(\mathbf{x})$ that *requires* 2^n size algebraic circuits?
  - Valiant's VP≠VNP question *pin-points* this $f(\mathbf{x})$.
- To better understand this, let's start with a familiar polynomial--- Determinant.

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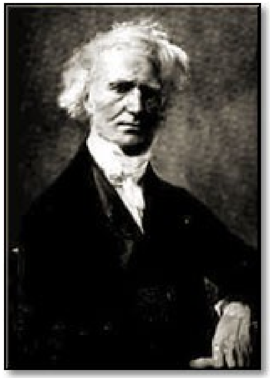
Warmup: Determinant

Augustin-Louis **Cauchy**
(1789--1857)

- Determinant is an $n=m^2$-variate m-degree polynomial defined as: eg. for $m=3$,

$$Det\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = x_{11}x_{22}x_{33} - x_{12}x_{21}x_{33} - x_{13}x_{22}x_{31} - x_{11}x_{23}x_{32} + x_{12}x_{23}x_{31} + x_{13}x_{21}x_{32}.$$



Jacques **Binet**
(1786--1856)

- It has $m*(m-1)*...*1 = m! \approx (m/e)^m \approx n^{\sqrt{n}/2}$ monomials.
- Qn: What's its circuit-size complexity ?
  - As large as the number of monomials??

Warmup: Determinant

- Theorem [Csanky 1976]: Det_m has a circuit of size $< m^7 < n^{3.5}$.
- How's this possible?
  - Algebraic ideas/ *identities* !
- *Idea 1:* $\mathrm{Det}_m(X) =$ product of the m eigenvalues of $X =: \prod \lambda_i$.
- *Idea 2:* $\prod\lambda_i$ is an expression in $p_k := \sum_{1\leqq i\leqq m} \lambda_i^k \quad (1\leqq k\leqq m)$.
  - Newton's identity.
- *Idea 3:* In turn, $p_k = \mathrm{tr}(X^k)$.
- Thus, $\mathrm{Det}(X)$ computation reduces to matrix-powering X^k.
  - Easy to implement in a circuit.

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Valiant's permanent

- Valiant (1977) posed a related polynomial– Permanent.

$$Per\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = x_{11}x_{22}x_{33}+x_{12}x_{21}x_{33}+x_{13}x_{22}x_{31}+ x_{11}x_{23}x_{32}+x_{12}x_{23}x_{31}+x_{13}x_{21}x_{32} .$$

OPEN Qn: Given a matrix X compute $Per_m(X)$?

- Counts the *number of satisfying assignments* of a given boolean formula.
  - Or, *number of matchings* in a graph.

- Its computation solves NP-hard problems!

- *Valiant's VP≠VNP question*:
  Per_m *requires* 2^m size algebraic circuits?

Valiant's permanent

- Show that permanent *affords no ultra-clever* algebraic identities that help in circuit computation?
- *Classical algebra* is not developed well enough to answer this question.
  - Permanent, circuits are both recent constructs.
  - A theory specialized to size is missing.
- We need to study circuit identities---

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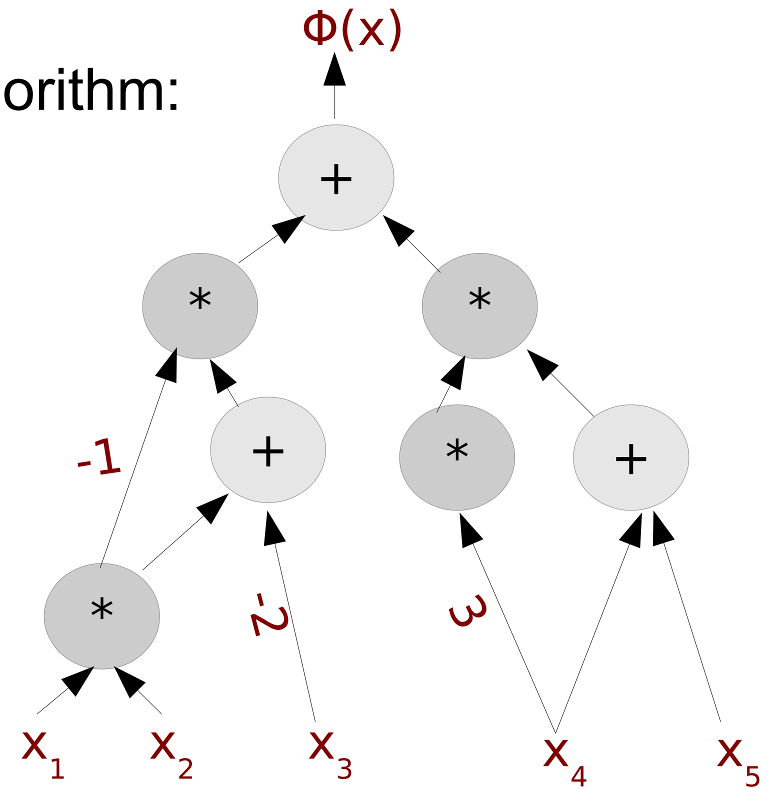
Zero or nonzero

- Question: Test whether a given circuit is *zero*.
  - Polynomial identity testing (PIT).
- PIT has a simple randomized fast algorithm:
  - Evaluate **Φ(p)** for a *random* point **p**.
- OPEN Qn: Is PIT in deterministic polynomial time?
- Work in last decades show:

<u>Meta-Theorem</u>: A solution of identity testing would answer the permanent question.



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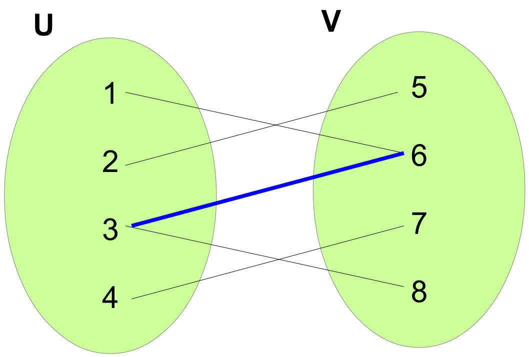
Application 1-- Graph theory

- Is there a perfect matching in a given graph?
- First solution by Dinitz (1970), Edmonds, Karp (1972).
  - Deterministic polynomial time using flows.
- Is there a fast parallel algorithm?
- Relates to PIT.
  - Write it as a determinant, circuit....



Application 2-- Number theory

- Circuits can compute high-degree polynomials.
  - Eg. $f(x) = x^{2^s}$ requires only circuit-size s.
  - Repeated squaring.

- Testing primality of n reduces to testing---

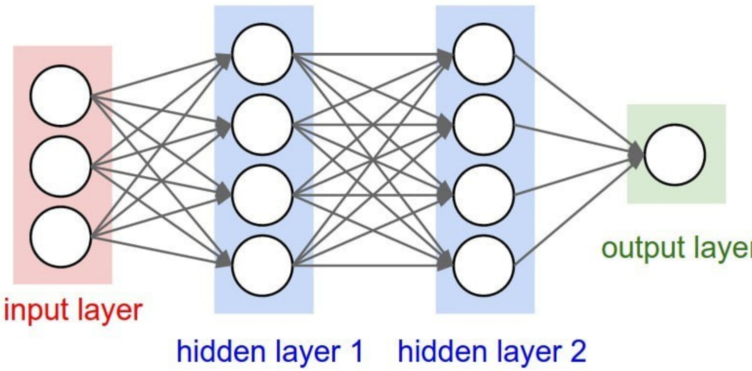
$$(x+1)^n = x^n + 1 \quad mod\ n\ .$$

- Used by Agrawal-Kayal-S. (2002) in primality testing.
  - First deterministic polynomial time primality test.

- Relates to derandomizing PIT.
  - Fix x to special values!

Application 3-- Learning theory

- Areas like *Artificial Intelligence / Machine Learning* model decision-making using circuits.
  - Artificial Neural Networks (ANN).



- ANN is a *specialized* algebraic circuit.
- Backpropagation methods are used to *modify the edges*, and their weights, to improve the output.
- Relates to better understanding of algebraic circuits?
  - Complexity of circuit learning problems...?

Thank you!