On the centers of higher degree forms

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Let $\Theta$ be a symmetric $d$-linear form on a vector space $V$ of dimension $n$ over a field $K$. Its center, $\text{Cent}(V, \Theta)$, is the analog of the space of symmetric matrices for a bilinear form. If $d > 2$, the center is a commutative subalgebra of $\text{End}_K(V)$. It was conjectured in [4] that the center has dimension at most $n$ and a proof was given for $n \leq 5$. We construct counter examples to this conjecture. We give an infinite family of cubic forms $\{(V_r, \Theta_r)\}_{r \geq 3}$ such that for any $\epsilon \in (0, 1)$ there exists $r_0(\epsilon)$ having the property:

$$\text{dim}_K \text{Cent}(V_r, \Theta_r) > (1 + \epsilon) \cdot \text{dim}_K V_r$$

1. Preliminaries

We first collect the definitions related to $d$-linear forms following [4]. Suppose $V$ is a vector space over a field $K$. A symmetric 2-linear form $\Theta : V \times V \to K$ has a geometrical interpretation as an inner product on $V$. The 2-linear form $\Theta$ induces a quadratic form $\phi : V \to K$ defined by:

$$\phi(v) = \Theta(v, v)$$

and conversely a quadratic form $\phi$ induces a 2-linear form: $2\Theta(u, v) = \phi(u + v) - \phi(u) - \phi(v)$ when char $K \neq 2$. These concepts can be generalized to higher dimensions $d$.

Definition 1.1. A $d$-linear space over $K$ is a pair $(V, \Theta)$ where $V$ is a finite dimensional $K$-vector space and $\Theta : V \times \cdots \times V \to K$ is a symmetric $d$-linear form. That is, $\Theta$ is $K$-linear in each of its slots and it is invariant under all permutations of the slots.

When char $K = 0$ or char $K > d$, these $d$-linear spaces are in one-one correspondence to $d$-homogeneous polynomials in $K[x_1, \ldots, x_n]$ where $n := \text{dim}_K V$. This can be seen by fixing a $K$-basis $e_1, \ldots, e_n$ of $V$ and defining $\phi(v) := \Theta(v, \ldots, v)$ where $v = x_1 e_1 + \cdots + x_n e_n$, then $\phi$ is a $d$-homogeneous polynomial in $K[x_1, \ldots, x_n]$.

Two $d$-linear spaces $(V, \Theta)$, $(V', \Theta')$ are isomorphic if there is an invertible linear map $t : V \to V'$ such that $\Theta'(tv_1, \ldots, tv_d) = \Theta(v_1, \ldots, v_d)$ for every $v_1, \ldots, v_d \in V$.

A notion of decomposability of $d$-linear spaces can be defined as follows:

Definition 1.2. The orthogonal sum $(V_1, \Theta_1) \perp (V_2, \Theta_2)$ of two spaces is the $d$-linear space on $V_1 \oplus V_2$ with form $\Theta_1 \perp \Theta_2$ defined as:

$$(\Theta_1 \perp \Theta_2)(u_1 + v_1, \ldots, u_d + v_d) := \Theta_1(u_1, \ldots, u_d) + \Theta_2(v_1, \ldots, v_d)$$
where, \( u_i \in V_1 \) and \( v_i \in V_2 \). A \( d \)-linear space \((V, \Theta)\) is decomposable if \((V, \Theta) \cong (V_1, \Theta_1) \perp (V_2, \Theta_2)\) for some nonzero spaces \((V_1, \Theta_1), (V_2, \Theta_2)\).

On the level of homogeneous polynomials the sum \( \Theta_1 \perp \Theta_2 \) corresponds to \((\phi_1 \perp \phi_2)(X_1, X_2) = \phi_1(X_1) + \phi_2(X_2)\) where \( X_1, X_2 \) are disjoint set of variables.

A \( d \)-homogeneous polynomial \( \phi \) whose number of variables cannot be reduced by a linear change of variables corresponds to a regular \( d \)-linear space \((V, \Theta)\).

**Definition 1.3.** \((V, \Theta)\) is said to be regular if \( \Theta(v, V, \ldots, V) = 0 \) implies \( v = 0 \). The expression \( \Theta(v, V, \ldots, V) = 0 \) is a shorthand for: \( \forall v_2, \ldots, v_d \in V, \Theta(v, v_2, \ldots, v_d) = 0. \)

The notion of symmetric matrices for bilinear forms generalizes to the center for higher dimensional forms.

**Definition 1.4.** The center \( \text{Cent}(V, \Theta) \) of a \( d \)-linear space \((V, \Theta)\) is defined as:

\[
\{ t \in \text{End}_K(V) \mid \Theta(tv_1, v_2, \ldots, v_d) = \Theta(v_1, tv_2, v_3, \ldots, v_d) \text{ for all } v_1, \ldots, v_d \in V \}\]

The following properties of the center were first proved in [2]:

**Lemma 1.5.** Suppose \((V, \Theta)\) is a regular \( d \)-linear space where \( d \geq 3 \).

1. \( \text{Cent}(V, \Theta) \) is a commutative \( K \)-subalgebra of \( \text{End}_K V \).
2. \((V, \Theta)\) is indecomposable if and only if \( \text{Cent}(V, \Theta) \) is local.

The following property of the center (see page 1277 of [3]) is useful in computing the structure. We provide the proof for the sake of completeness.

**Lemma 1.6.** Let \((V, \Theta)\) be a \( d \)-linear space and let \( n := \text{dim}_K V \). Then

\[
\text{Cent}(V, \Theta) \cong \{ M \in K^{n \times n} \mid (JM)^T = JM \}
\]

where \( J = \left( \frac{\partial^2 f}{\partial x_i \partial x_j} \right) \) is the Hessian matrix of the \( d \)-homogeneous polynomial \( f(x_1, \ldots, x_n) \) corresponding to \((V, \Theta)\).

**Proof.** Let us fix a \( K \)-basis \( \{ e_i \}_{1 \leq i \leq n} \) of \( V \) such that \( e_i \) is an \( n \times 1 \) vector having all zeros except a 1 at the \( i \)-th position. Suppose \( t \in \text{Cent}(V, \Theta) \) then by definition for all \( 1 \leq i, j \leq n \):

\[
\forall v_3, \ldots, v_d \in V, \Theta(te_i, e_j, v_3, \ldots, v_d) = \Theta(e_i, te_j, v_3, \ldots, v_d) \quad (1)
\]

Let \( M \) be a matrix whose \( i \)-th column is equal to \( te_i \), for all \( 1 \leq i \leq n \). Thus, LHS of equation (1) is

\[
\Theta\left( \left( \sum_{l=1}^{n} M_l e_l \right), e_j, v_3, \ldots, v_d \right) = \sum_{l=1}^{n} M_l \Theta(e_l, e_j, v_3, \ldots, v_d)
\]
Now \( \Theta(e_1, e_j, v_3, \ldots, v_d) \) is a \((d - 2)\)-linear form treating the first 2 arguments fixed and the last \((d - 2)\) taking values from \( V \). It can be easily verified that this \((d - 2)\)-linear form corresponds to the \((d - 2)\)-homogeneous polynomial:

\[
\frac{1}{d(d-1)} \cdot \frac{\partial^2 f}{\partial x_i \partial x_j}
\]

Thus, the LHS of equation (1) corresponds to the following \((d - 2)\)-homogeneous polynomial:

\[
\frac{1}{d(d-1)} \cdot \sum_{i=1}^{n} M_{li} \frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{1}{d(d-1)} \cdot \sum_{i=1}^{n} M_{li} J_{ji} = \frac{1}{d(d-1)} \cdot (JM)_{ji}
\]

Similarly, the RHS of equation (1) corresponds to \( \frac{1}{d(d-1)} \cdot (JM)_{ij} \). Since equation (1) holds for all \( i, j \) we get \( (JM)^T = JM \).

It was conjectured in [4] that if \( d \geq 3 \) and \((V, \Theta)\) is a regular \( d \)-linear space over the field \( K \) then \( \dim_K \text{Cent}(V, \Theta) \leq \dim_K V \). In this paper we construct an infinite family of cubic forms \( \text{i.e. for } d = 3 \) which are exceptions to the conjecture. In these counter examples the lower bound on the fraction \( \frac{\dim_K \text{Cent}(V, \Theta)}{\dim_K V} \) can be made arbitrarily close to 2.

2. The Counter Example

The following theorem summarizes the construction of the counter examples.

**Theorem 2.1.** Let \( r \geq 3 \). Consider the cubic polynomial:

\[
f(z, \bar{b}) := \sum_{1 \leq i \leq j \leq r} z_{ij} b_i b_j .
\]

Let \((V, \Theta)\) be the 3-linear space corresponding to \( f \) over a field \( K \) with \( \text{char}(K) \neq 2, 3 \). Then

1. \((V, \Theta)\) is regular and indecomposable.

2. \( n := \dim_K V = r + \frac{r(r+1)}{2} \) and \( \dim_K \text{Cent}(V, \Theta) \geq r^2 + 1 \). Thus,

\[
\frac{\dim_K \text{Cent}(V, \Theta)}{\dim_K V} \geq 2 - \frac{6r - 2}{r^2 + 3r} > 1 .
\]

**Proof.** Let \( s := \frac{r(r+1)}{2} \). We will use lemma 1.6 to compute the structure of \( \text{Cent}(V, \Theta) \). As in the proof of lemma 1.6, let \( J = \left( \frac{\partial^2 f}{\partial x_i \partial x_j} \right) \) be the Hessian matrix of \( f \). Fix a \( K \)-basis of \( V \) as \( e_i = (\text{an } n \times 1 \text{ vector with } i-\text{th entry } 1 \text{ and the rest zeros}) \) and for any \( t \in \text{End}_K(V) \) define a corresponding matrix \( M \) whose \( i \)-th column is \( te_i \) for all \( 1 \leq i \leq n \). Note that the following relation holds: for any \( v \in V \), \( tv = Mv \). We will show that if \( t \in \text{Cent}(V, \Theta) \) then \( M \) is of a very special form:
Claim 2.1.1. If \( t \in \text{Cent}(V, \Theta) \) then for some \( c \in K, \ A \in K^{s \times r} \)

\[
M = cI + \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}.
\]

Consequently, \((V, \Theta)\) is indecomposable and \( \text{Cent}(V, \Theta) \cong K \oplus \mathcal{N} \) where \( \mathcal{N} \) is a commutative nil algebra in which product of any two elements vanish.

Proof of Claim 2.1.1. From the definition, \( t \in \text{Cent}(V, \Theta) \) iff

\[
\Theta \left( M \begin{pmatrix} z_{1,*} \\ b_{1,*} \end{pmatrix}, \begin{pmatrix} z_{2,*} \\ b_{2,*} \end{pmatrix} \right) = \Theta \left( \begin{pmatrix} z_{1,*} \\ b_{1,*} \end{pmatrix}, M \begin{pmatrix} z_{2,*} \\ b_{2,*} \end{pmatrix} \right)
\]

where for all \( 1 \leq i \leq 3 \), \( z_{i,*} \) represents the column vector (having \( s \) entries):

\[
\begin{pmatrix} z_{i,1,1} & z_{i,1,2} & \cdots & z_{i,r-1,r} & z_{i,r,r} \end{pmatrix}^T
\]

and similarly \( b_{i,*} \) represents the column vector (having \( r \) entries):

\[
\begin{pmatrix} b_{i,1} & \cdots & b_{i,r} \end{pmatrix}^T
\]

Remember that \( \Theta \) is the natural 3-linear form obtained from \( f \), so we intend to associate the \( z_{i,j,k} \) component to the \( z_{j,k} \) variable of \( f \) and the \( b_{i,j} \) component to the \( b_{j} \) variable. In equation form this yields:

\[
\Theta(v, v, v) = f(\vec{\tau}, \vec{b}) = \sum_{1 \leq i \leq j \leq r} z_{i,j} b_i b_j
\]

where \( v := z_{1,1}e_1 + z_{1,2}e_2 + \ldots + z_{r-1,r}e_{s-1} + z_{r,r}e_s + b_1e_{s+1} + \ldots + b_re_n \).

Let us compare the coefficient of \( z_{3,i,i} \) on both sides of equation (2) to get:

\[
\frac{1}{3} \tau(b_{1,i})b_{2,i} = \frac{1}{3} b_{1,i} \tau(b_{2,i}) \quad \text{where} \quad \tau(.) \text{ is the effect of } M.
\]

As this equation holds for all values of \( z_{1,*}, b_{1,*}, z_{2,*}, b_{2,*} \) we obtain that \( \tau(b_{1,i}) = c_i b_{1,i} \) for some \( c_i \in K \). Now compare the coefficients of \( z_{3,i,j} \) for \( i < j \) on both sides of equation (2):

\[
\frac{1}{6} (\tau(b_{1,i})b_{2,j} + \tau(b_{1,j})b_{2,i}) = \frac{1}{6} (b_{1,i} \tau(b_{2,j}) + b_{1,j} \tau(b_{2,i}))
\]

\[
\Rightarrow c_i b_{1,i} b_{2,j} + c_j b_{1,j} b_{2,i} = c_j b_{1,i} b_{2,j} + c_i b_{1,j} b_{2,i} .
\]

This forces \( c_i = c_j \) and hence \( c_1 = \cdots = c_r =: c \in K \). Thus, the last \( r \) rows of \((M - cI)\) are zero.

Note that equation (2) holds if we substitute \((M - cI)\) instead of \( M \). We will keep using \( \tau(.) \) as the effect of \((M - cI)\). Let us compare coefficients of \( b_{3,j} \) on both sides of this modified form of equation (2):

\[
\frac{1}{3} (\tau(z_{1,i,j})b_{2,j} + z_{2,j,i} \tau(b_{1,i})) + \frac{1}{6} \sum_{i \neq j} \tau(z_{i,i,j}) b_{2,i} + z_{2,i,j} \tau(b_{1,i})
\]
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\[ \frac{1}{3} (\tau(z_{2,j,j})b_{1,j} + z_{1,j,j}\tau(b_{2,j})) + \frac{1}{6} \sum_{i=1}^{r} (\tau(z_{2,i,j})b_{1,i} + z_{1,i,j}\tau(b_{2,i})) \cdot \]

This expression can be considerably simplified by observing that the last \( r \) rows of \((M-cI)\) are zero and hence \( \tau(b_{*,*}) = 0 \):

\[ \frac{1}{3} \tau(z_{1,j,j})b_{2,j} + \frac{1}{6} \sum_{i=1}^{r} \tau(z_{1,i,j})b_{2,i} = \frac{1}{3} \tau(z_{2,j,j})b_{1,j} + \frac{1}{6} \sum_{i=1}^{r} \tau(z_{2,i,j})b_{1,i} \cdot \]

Again, as this equation holds for all values of \( z_{1,*}, b_{1,*}, z_{2,*}, b_{2,*} \) we deduce that \( \tau(z_{1,i,j}) \) is only a linear combination of \( b_{1,k} \)'s and has no \( z_{1,*} \). Thus,

\[ (M-cI) = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}_{n \times n} \text{ where } A \in K^{s \times r} \]
and $v_3 = (a \text{ column vector having all zeros except } b_{3,j} = 1)$. Since $\Theta$ is induced by $f$ we obtain:

$$\Theta(v_1, v_2, v_3) = \frac{1}{3} z_{1,i,j} \text{ or } \frac{1}{6} z_{1,i,j} \text{ depending on whether } i = j \text{ or not.}$$

If $b_{1,i} \neq 0$ then define $v_2 = (a \text{ column vector having all zeros except } b_{2,i} = 1)$ and $v_3 = (a \text{ column vector having all zeros except } z_{3,i,i} = 1)$. Again we get:

$$\Theta(v_1, v_2, v_3) = \frac{1}{3} b_{1,i}$$

Thus, for all nonzero $v_1 \in V$ one of the above two definitions will give $v_2, v_3 \in V$ such that $\Theta(v_1, v_2, v_3) \neq 0$. By definition this means that $(V, \Theta)$ is regular.

This completes the description of the counter example.

Interestingly, cubic forms like $f$ can capture the graph isomorphism problem. That is, given two (finite) graphs $G_1, G_2$ we can effectively construct two cubic forms $f_{G_1}, f_{G_2}$ such that:

$$G_1 \cong G_2 \iff f_{G_1} \cong f_{G_2}$$

For further details see sections 6 and 7 of [1].

Finally, we would like to pose the following question: Given a $d$-linear space $(V, \Theta)$ (where $d > 2$)

is $\dim_K \text{Cent}(V, \Theta) \leq (d - 1) \cdot \dim_K V$?

Acknowledgement

I would like to thank Susanne Pumpluen for many useful email discussions and for reviewing a preliminary version of the paper.

REFERENCES


