An effective description of roots of bivariates mod p^k and the related Igusa's local zeta function

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Algebra over $\mathbb{Z}\ /\boldsymbol{p^k}\mathbb{Z}$

- Ring: mod p^k.
- Operations: $a + b = a + b \bmod p^k$; $a \times b = a \cdot b \bmod p^k$.
- Elements: $a_0 + a_1 p + \cdots + a_{k-1} p^{k-1}$; $\quad a_i \in \{0, \ldots, p-1\}$.
- $(\mathbb{F}_p, \mathbb{F}_p, \ldots, \mathbb{F}_p)$.

Algebra over $\mathbb{Z}\ /\boldsymbol{p^k}\mathbb{Z}$

- $f(x) = x^2 + px \bmod p^2$.
- How many factors?
- Roots: $0, p, 2p, \ldots, (p-1) \cdot p$.
- $(x - i.p) \mid (x^2 + px) \mod p^2$.
- $x^2 + px = (x+0)(x+1.p) = \cdots = (x+i.p)(x+(p-i+1).p)$.
- Not Unique Factorization Domain (UFD).

Algebra over $\mathbb{Z}/\boldsymbol{p}^{\boldsymbol{k}}\mathbb{Z}$

- Find roots modulo p^k
- Root-finding over any commutative ring: NP-complete
- Given $f(\mathbf{x}) \in \boldsymbol{R}[\mathbf{x}]$, find a root of $f(\mathbf{x})$ over $\boldsymbol{R}$.
- Over $\mathbb{F}_p$: [HW99,LPTWY17].
- Difficult over $\mathbb{Z}/p^k\mathbb{Z}$: Lifting.

Notations

- Effective polynomial: Polynomial modulo p.
- Effective degree: Degree of effective polynomial.

  E.g., $f(x) = x^2 + px^3$ has effective polynomial x^2 and effective degree 2.
- Valuation: $v_p(a) = v$ such that $p^v | a$ but $p^{v+1} \nmid a$.
- Val-multiplicity: $v_p(f(a + px))$.

Lifting of roots

- Elements: $a_0 + a_1 p + \cdots + a_{k-1} p^{k-1}$; $a_i \in \{0, \ldots, p-1\}$
- Roots mod p^k $\Longrightarrow$ roots mod p^{k-1}.
- Roots mod p^{k-1} $\Longrightarrow$?? roots mod p^k.

Lifting of roots

- Roots mod p^{k-1} → $??$ roots mod p^k.
- Elements: $a_0 + a_1 p + \cdots + a_{k-1} p^{k-1}$; $a_i \in \{0, \ldots, p-1\}$
- Find each $\mathbb{F}_p$-coordinate separately.

- $f(x) \bmod p$ → root a_0.
  Let $\tilde{f}(x) := p^{-v} f(a_0 + px)$ → root a_1.
- Root: $(a_0 + a_1 p)$ of $f(x) \bmod p^2$.

} Lifting of roots

Lifting of roots

Example:

- Let $f(x) = x^3 - x^2 + p \bmod p^2$.
- $f(x) \bmod p = x^3 - x^2$ ⟹ roots $\{0, 1\}$.

Lifting of roots

- First coordinate: 0
  - Roots of $x^3 - x^2 + p \bmod p^2$ of the form: $0 + px$.
  - Roots of $(0 + px)^3 - (0 + px)^2 + p \bmod p^2$.
  - Roots of $p(p^2x^3 - px^2 + 1) \bmod p^2$.
  - Roots of $(p^2x^3 - px^2 + 1) \bmod p$.
  - None exist (0 does not lift)!

Lifting of roots

- First coordinate: 1
  - Roots of $x^3 - x^2 + p \bmod p^2$ of the form: $1 + px$.
  - Roots of $(1 + px)^3 - (1 + px)^2 + p \bmod p^2$.
  - Roots of $p(p^2x^3 + 2px^2 + x + 1) \bmod p^2$.
  - Roots of $(p^2x^3 + 2px^2 + x + 1) \bmod p$.
  - Root exists (1 does lift)!
  - Required root $1 + (p - 1)p$.

Univariate root-finding [BLQ13]

- Algorithm root-find($f(x), p^k$):
  1. If $k = 0$, return $*$.
  2. If $\deg(f) = 1$, return the root.
  3. Factorize $f(x) \bmod p$. Root set $=: S$.
  4. For each $a \in S$:
     1. Find roots of root-find($p^{-v} f(a + px), p^{k-v}$).

- Representative roots:
  - Roots of above algorithm of the form $a_0 + a_1 p + \cdots + a_{k_1} p^{k_1} + p^{k_1+1} *$.
  - p^{k-k_1-1} many roots.

Lifting of roots: Hensel's lifting

Theorem: Effective polynomial is linear ⇒ roots mod p^k exist for every k.

Example: $f(x_1, x_2) = \ell x_1 + m x_2 + n + p.\, g(x_1, x_2)$ has root (a_1, a_2).

Lifting: $\underbrace{\ell a_1 + m a_2 + n}_{\text{Divisible by } p} + \underbrace{p \ell x_1 + p n x_2}_{\text{val-mult = 1}} + \underbrace{p g(a_1 + p x_1, a_2 + p x_2)}_{p.\,C + p^2.\,h(x_1, x_2)}$.

Continues to stay linear!

Linear-representative roots

Structure of polynomial after lifting

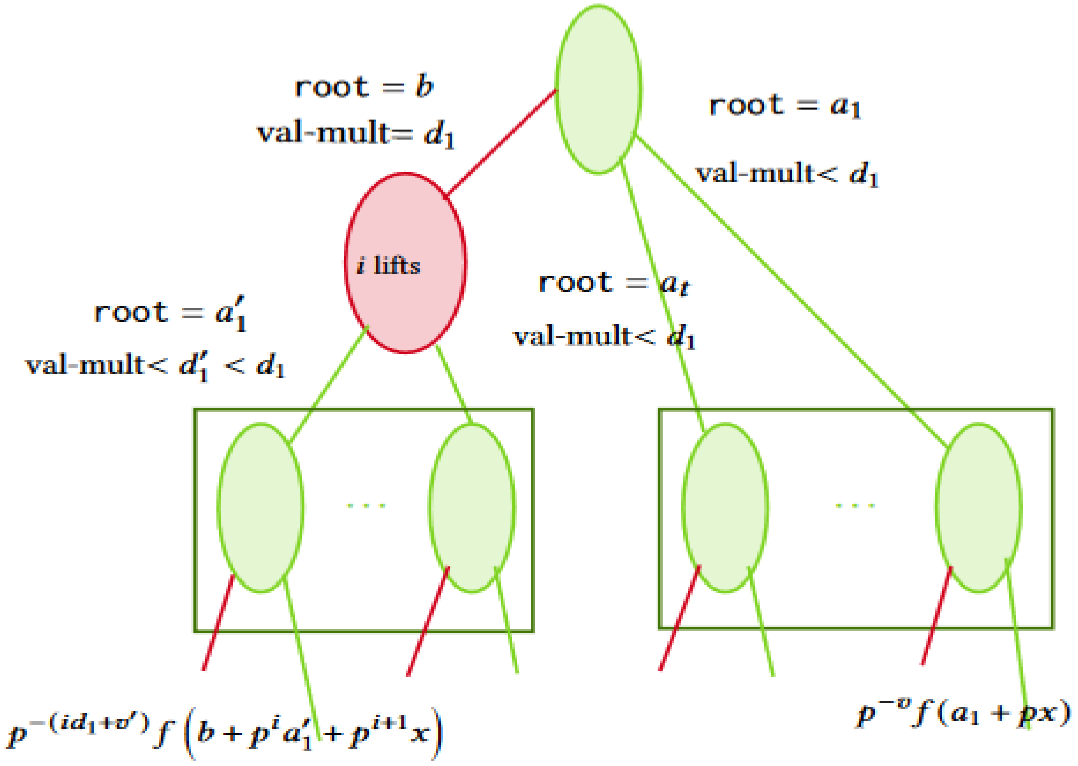
- Effective degree change?
- $f(x_1, x_2) \rightarrow \tilde{f}(x_1, x_2) := p^{-v} f(a_1 + px_1, a_2 + px_2)$.
- Effective degree of $f(x_1, x_2) \rightarrow d_1$.
  Effective degree of $\tilde{f}(x_1, x_2) \rightarrow d_2$.

- **Theorem:** Effective degree reduces with lifting: $d_2 \leq v \leq d_1$.

- **Goal:** Reduce to linear representative roots **OR** achieve power of p.

$v^{(1)} + \cdots + v^{(t)} \geq k$

Search for roots

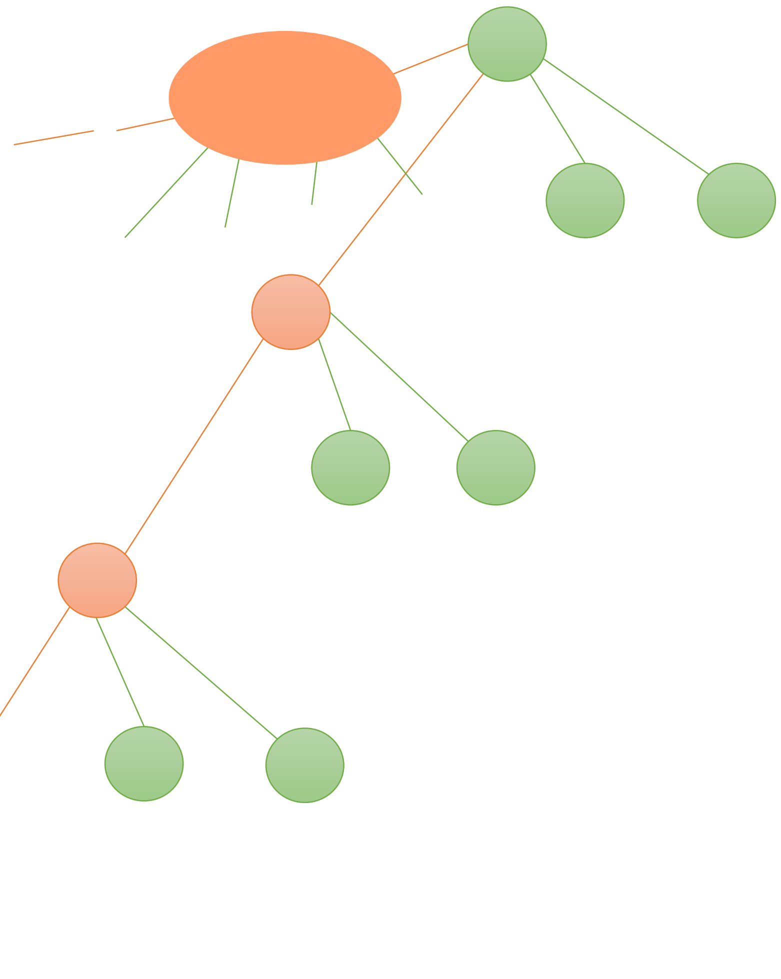


- Degree reduction case
- Constant degree case

Nodes: polynomials
Branches: local roots
Leaves: linear-representatives

Poly in p^d?

Search for roots



Constant degree case

- Val-multiplicity d_1 roots:
  - Unique val-multiplicity d_1 roots

    d_1-nonpower form: $\langle x_1 - a_1, x_2 - a_2 \rangle^{d_1}$.
  - Multiple val-multiplicity d_1 roots

    d_1-power: $(a_1 x_2 - a_2 x_1)^{d_1}$.

Reduction to univariates

- For d_1-nonpower form:
  - Only one branch exists.
- For d_1-power:
  - Required structure $(a_2x_1 - a_1x_2)^{d_1}$ at each of i_1-steps.
  - Let $g(L, x_2) = f(x_1, x_2)$.
  - Write $g(L, x_2)$ as $L^{d_1} + L^{d_1-1} \cdot u_1(x_2) + L^{d_1-2} \cdot \frac{u_2(x_2)}{p} + \cdots + \frac{u_{d_1}(x_2)}{p^{d_1-1}}$.
  - Require $\left(L + \frac{u_1(x_2)}{d_1}\right)^{d_1} \bmod p$ to lift using d_1-power again.
  - System of equations $u_j(x_2) = p^{j-1} \cdot {}^{d_1}C_j \cdot \left(\frac{u_1(x_2)}{d_1}\right)^j \bmod p^j$.

Chain of val-mult$= \boldsymbol{d_1}$ roots

- Contiguous chain of 'i_1' d_1-powers and 'i_3' d_1-nonpower forms .
- $i_1 + i_3 \leq k/d_1$.
- Iterate over all possible i_1, i_3.

Moving to $\boldsymbol{p}$-adics

- $\mathbb{Z}\,/p^k\mathbb{Z}$ where $k \to \infty$.
- Elements: $a_0 + a_1 p + a_2 p^2 + \cdots$; $a_i \in \{0, \ldots, p-1\}$.
- Different from integers?
- Not countable.
- $f(x) = x^2 + 1, \quad p = 2$.
- Root over $\mathbb{Z}_2$ but not $\mathbb{Z}$.

Moving to $\boldsymbol{p}$-adics

- k → how large?
- Roots over $\mathbb{Z}_p$ from:
  - Linear representative roots.
  - Blowing up of 0: homogeneous polynomial, e.g., $x_1^2 + (p+1)x_1x_2$.
- $k_0 = O(d^{10}\log M)$.

Igusa's local zeta function

- Given $f(\mathbf{x}) \in \mathbb{Z}_p[x_1, \ldots, x_n]$.
- Define IZF: $Z_{f,p}(s) := \int_{\mathbf{Z}_p^n} |f(\mathbf{x})|_p^s \cdot |d\mathbf{x}|$.

  where $s \in \mathbb{C}$ and $\mathrm{Re}(s) > 0$.
- Poincare series: $P_{f,p}(t) := \sum_{i=0}^{\infty} N_{p^i}(f) \cdot (p^{-n}\, t)^i$,

  where $t \in \mathbb{C}$ with $|t| < 1$, $N_{p^i}(f)$:= # roots of f mod p^i.
- $P(t) = \frac{1 - t \cdot Z_{f,p}(s)}{1 - t}$

  where $t = p^{-s}$.

Igusa’s local zeta function

- Hardness: #-P hard.
- Proof of rationality: [Igu74,Igu77,Den84].
- Efficient algorithms for computing IZF: [ZG03,DS20] (univariates).

Igusa's LZF: Counting roots modulo every $\boldsymbol{p^k}$

- Constant k -> $N_k(f)$ is constant.
- Non-constant?
- $N_k(f)$ for all $k \ < \ k_0$.
- $N_k(f)$ for $k \ > \ k_0$:
  1. Linear-representative roots mod $p^{k_0} \Rightarrow p^{k-k_0}$.
  2. Blowing up of zero roots after $k_0 \Rightarrow (k - k_0)p^{k-k_0}$.

Thank you!

Summary

- Roots: Linear-rep. roots/ non-linear rep. roots
- Root finding: Degree reduction
- Constant degree: Reduction to univariates
- k_0: Gap between $\mathbb{Z}_p$ and mod p^k
- $\mathbb{Z}_p$ roots
- Rationality of IZF for bivariates

Future work

- n-variates
- Root finding in polylog(p) $(d, n$ const$)$