From Sylvester-Gallai Configurations to Rank Bounds: Improved Black-box Identity Test for Depth-3 Circuits

Nitin Saxena
Hausdorff Center for Mathematics
Bonn - 53115, Germany
ns@hcm.uni-bonn.de

C. Seshadhri
IBM Almaden
San Jose - 95126, USA
csesha@us.ibm.com

***Abstract*—We study the problem of identity testing for depth-3 circuits of top fanin k and degree d. We give a new structure theorem for such identities. A direct application of our theorem improves the known deterministic $d^{k^{O(k)}}$-time black-box identity test over rationals (Kayal & Saraf, FOCS 2009) to one that takes $d^{O(k^2)}$-time. Our structure theorem essentially says that the number of independent variables in a real depth-3 identity is very small. This theorem affirmatively settles the strong rank conjecture posed by Dvir & Shpilka (STOC 2005).**

We devise a powerful algebraic framework and develop tools to study depth-3 identities. We use these tools to show that any depth-3 identity contains a much smaller *nucleus identity* that contains most of the "complexity" of the main identity. The special properties of this nucleus allow us to get almost optimal rank bounds for depth-3 identities.

***Keywords*-depth-3 circuit; identities; Sylvester-Gallai; incidence configuration; Chinese remaindering; ideal theory**

I. Introduction

Polynomial identity testing (PIT) ranks as one of the most important open problems in the intersection of algebra and computer science. We are provided an arithmetic circuit that computes a polynomial $p(x_1, x_2, \cdots, x_n)$ over a field $\mathbb{F}$, and we wish to test if p is identically zero (in other words, if p is the zero polynomial). In the black-box setting, we do not have access to the circuit. We are only allowed to evaluate the polynomial p at various domain points. The main goal is to devise a *deterministic* (preferably black-box) polynomial time algorithm for PIT. Heintz & Schnorr [HS80], Kabanets & Impagliazzo [KI04] and Agrawal [Agr05], [Agr06] have shown connections between deterministic algorithms for identity testing and circuit lower bounds, emphasizing the importance of this problem. For a detailed exposition, see surveys [Sax09], [AS09].

Even for the special case of depth-3 circuits, this question is still open. This may seem quite depressing. It is. Nonetheless, there exist concrete results that justify both our ignorance and the acceptance of results on depth-3 PIT in major publishing venues. Agrawal and Vinay [AV08] showed that an efficient black-box identity test for depth-4 essentially leads to subexponential lower bounds.

A depth-3 circuit C over a field $\mathbb{F}$ is of the form $C(x_1, \cdots, x_n) = \sum_{i=1}^{k} T_i$, where T_i (*a multiplication term*) is a product of at most d linear polynomials with coefficients in $\mathbb{F}$. We are especially interested in the case $\mathbb{F} = \mathbb{Q}$. In this section, we will just assume this unless explicitly mentioned otherwise. The size of the circuit C can be expressed in three parameters: the number of variables n, the degree d, and the *top fanin* (or the number of terms) k. Such a circuit is referred to as a $\Sigma\Pi\Sigma(k, d, n)$ circuit. PIT algorithms for depth-3 circuits were first studied by Dvir & Shpilka [DS06]. There have been many recent results in this area by Kayal & Saxena [KS07] (in the non-black-box setting), Karnin & Shpilka [KS08], Saxena & Seshadhri [SS09], and Kayal & Saraf [KS09b]. Our main result is a better black-box tester for $\Sigma\Pi\Sigma$ circuits over $\mathbb{Q}$. We get a running time of nd^{k^2}, an exponential improvement (in k) over the previous best of nd^{k^k} [KS09b]. Table I details the time complexities of previous algorithms. These time complexities are actually bounds on the total number of bit operations. Also, the running times are technically polynomial in the stated times.

Theorem 1: Consider circuits over $\mathbb{Q}$. There exists a deterministic black-box algorithm for PIT on $\Sigma\Pi\Sigma(k, d, n)$ circuits, whose time complexity is poly(nd^{k^2}).

Table I: Depth-3 Black-box PIT algorithms over $\mathbb{Q}$

Paper	Time complexity
[KS08]	$nd^{(2^{k^2} \log^{k-2} d)}$
[SS09]	$nd^{k^3 \log d}$
[KS09b]	$nd^{(k^k)}$
This paper	nd^{k^2}

This is the first result that gives a time complexity both polynomial in d and singly-exponential in k for $\mathbb{Q}$. This is not too far from the best *non-black-box* algorithm for $\Sigma\Pi\Sigma$ circuits, which runs in poly(nd^k) time [KS07]. This result closes the gap (almost) between black-box and non-black-box algorithms.

All these results go via *rank bounds for depth-3 identities,* introduced by Dvir & Shpilka [DS06]. This is a very interesting quantity associated with these circuits, and roughly speaking, bounds the maximum number of "free variables" that can be present in a depth-3 identity. If a $\Sigma\Pi\Sigma(k, d, n)$ circuit has rank r, then there exists a linear transformation that converts this to an equivalent $\Sigma\Pi\Sigma(k, d, r)$ circuit. (This linear transformation is very easy to determine.) The remarkable insight of [DS06] was that the rank of *every* $\Sigma\Pi\Sigma(k, d, n)$ identity is very low. Any $\Sigma\Pi\Sigma(k, d, r)$-circuit can be completely expanded out in poly(kd^r) time. Hence, low rank bounds for identities imply efficient non-black-box PIT algorithms.

Karnin & Shpilka [KS08] showed how small rank bounds for identities imply efficient *black-box* PIT algorithms. This opened the door for black-box algorithms for depth-3 PIT. Indeed, all known algorithms for this problem come as a consequence of their result. Rank bounds have also found applications in learning $\Sigma\Pi\Sigma$ circuits [Shp09], [KS09a]. Hence, the rank and file of researchers studying this problem are interested in proving small rank bounds. As mentioned earlier, we focus on the field $\mathbb{Q}$. Dvir & Shpilka [DS06] initiated this line of work by showing that the rank of a simple, minimal $\Sigma\Pi\Sigma(k, d)$ identity is $2^{O(k^2)}(\log d)^{k-2}$. There are basic constructions of rank $\Omega(k)$ identities over $\mathbb{Q}$ [DS06]. Dvir & Shpilka [DS06] conjectured that the rank should be bounded by poly(k). This rank bound was improved to $O(k^3 \log d)$ by Saxena & Seshadhri [SS09]. Kayal & Saraf [KS09b] achieved a breakthrough by proving a rank bound independent of d. Their bound was $k^{O(k)}$. We finally settle the Dvir-Shpilka conjecture and show a rank bound of $O(k^2)$.

The advances of Kayal & Saraf were obtained through the use of *incidence geometry theorems*, like the famous Sylvester-Gallai theorem. This theorem states that for any set S of points in the Euclidean plane, not all collinear, there exists a line passing through exactly two points in S. Generalizations of this to higher dimensions are called *Sylvester-Gallai theorems* (see survey [BM90]). This theorem and its generalizations have connections to rank bounds for depth-3 circuits. The result of [KS09b] gave an intricate combinatorial construction that converts depth-3 identities to sets of colored points in Euclidean space. This allowed the use of Sylvester-Gallai theorems to bound the rank.

Our contribution comes through a new *algebraic* framework for studying depth-3 identities. This has many benefits. Firstly, it allows for a much more "efficient" use of Sylvester-Gallai theorems to bound the rank. This leads to nearly optimal rank bounds. Secondly, the connection between Sylvester-Gallai theorems and rank bounds is far more transparent, at the loss of some color from the theorems. Theorem 4 gives a simple formula that relates the depth-3 rank to Sylvester-Gallai bounds. A nice byproduct of this connection is the improvement of rank bounds over arbitrary fields. Thirdly, we get a deep structural theorem about depth-3 identities over any field. Every such identity contains a *nucleus identity* expressed on few variables. This nucleus, in some sense, captures all the complexity of the original identity, and has some very special properties. A better understanding of this nucleus may lead us to the goal of a truly polynomial time algorithm.

A. *Definitions and results*

We recall that a depth-3 circuit C over a field $\mathbb{F}$ is: $C(x_1, \ldots, x_n) = \sum_{i=1}^{k} T_i$, where T_i is a product of d_i linear polynomials $\ell_{i,j}$ over $\mathbb{F}$. For the purposes of studying identities we can assume, by homogenization, that $\ell_{i,j}$'s are linear *forms* (i.e. linear polynomials with a zero constant coefficient) and $\forall i, d_i = d$. It will be convenient to state our results in terms of arbitrary fields.

Definition 2: [DS06]

- **Simple Circuit:** C is a *simple* circuit if there is no nonzero linear form dividing all the T_i's.
- **Minimal Circuit:** C is a *minimal* circuit if for every proper subset $S \subset [k]$, $\sum_{i \in S} T_i$ is nonzero.
- **Rank of a circuit:** The coefficients of $\ell_{i,j}$ form an n-dimensional vector over $\mathbb{F}$. The *rank* of the circuit, rk(C), is defined as the rank of the set of all linear forms $\ell_{i,j}$ viewed as vectors.

The rank of a circuit can be interpreted as the minimum number of independent variables required to express C. The definition of simple and minimal circuits are used to remove certain pathological cases. The rank question is: for a simple and minimal $\Sigma\Pi\Sigma(k, d, n)$ identity over field $\mathbb{F}$, what is the maximal possible rank? A trivial upper bound on the rank (for any $\Sigma\Pi\Sigma$-circuit) is kd, since that is the total number of linear forms involved in C. A substantially smaller rank bound than kd shows that identities do not have as many "degrees of freedom" as general circuits.

Before we state our results, it will be helpful to explain *Sylvester-Gallai configurations*. A set of points S with the property that every line through two points of S passes through a third point in S is called a *Sylvester-Gallai configuration*. The famous Sylvester-Gallai theorem states that the only Sylvester-Gallai configuration in $\mathbb{R}^2$ is a set of collinear points. This basic theorem about point-line incidences was extended to higher dimensions [Han65], [BE67]. We define the notion of *Sylvester-Gallai rank bounds*. This is a clean and convenient way of expressing these theorems.

Definition 3: Let S be a finite subset of the *projective space* $\mathbb{FP}^n$. Alternately, S is a set of non-zero vectors in $\mathbb{F}^{n+1}$ without *multiples*: no two vectors in S are scalar multiples of each other. Suppose, for every set $V \subset S$ of k linearly independent vectors, the linear span of V contains at least $k+1$ vectors of S. Then, the set S is said to be *SG$_k$-closed*.

The largest possible rank of an SG_k-closed set of at most m vectors in $\mathbb{F}^n$ (for any n) is denoted by $\mathrm{SG}_k(\mathbb{F}, m)$.

The Sylvester-Gallai theorem states Higher dimensional analogues [Han65], [BE67] can be interpreted to say $\mathrm{SG}_k(\mathbb{R}, m) \leq 2(k-1)$. Our main theorem is a simple, clean expression of how Sylvester-Gallai influences identities. We state this for general fields.

Theorem 4 (From SG_k to Rank): Let $|\mathbb{F}| > d$. The rank of a simple and minimal $\Sigma\Pi\Sigma(k, d)$ identity over $\mathbb{F}$ is at most $2k^2 + k \cdot \mathrm{SG}_k(\mathbb{F}, d)$.

A direct application of the $SG_k(\mathbb{R}, m)$ bound yields an almost optimal rank bound for real depth-3 identities. For completeness, we state the exact rank bound obtained. We have a slightly stronger version (Theorem 18) of the above theorem that gives better constants.

Theorem 5 (Depth-3 Rank Bounds): Let C be a $\Sigma\Pi\Sigma(k, d)$ circuit, over field $\mathbb{R}$, that is simple, minimal and zero. Then, $\mathrm{rk}(C) < 3k^2$.

As discussed before, an application of this result to Lemma 4.10 of [KS08] gives a deterministic black-box identity test for $\Sigma\Pi\Sigma(k, d, n)$ circuits over $\mathbb{Q}$. Formally, we get the following *hitting set generator* for $\Sigma\Pi\Sigma$ circuits with real coefficients.

Corollary 6 (Black-box PIT over $\mathbb{Q}$): There is a deterministic algorithm that takes as input a triple (k, d, n) of natural numbers and in time $\mathrm{poly}(nd^{k^2})$, outputs a hitting set $\mathcal{H} \subset \mathbb{Z}^n$ with the following properties:

1) Any $\Sigma\Pi\Sigma(k, d, n)$ circuit C over $\mathbb{R}$ computes the zero polynomial iff $\forall a \in \mathcal{H}$, $C(a) = 0$.
2) $\mathcal{H}$ has at most $\mathrm{poly}(nd^{k^2})$ points.
3) The total bit-length of each point in $\mathcal{H}$ is $\mathrm{poly}(kn \log d)$.

1) Other fields: What about other fields? The rank bounds of [DS06] and [SS09] hold for arbitrary fields, whereas the rank bound of [KS09b] holds only for $\mathbb{R}$. It has been observed that for finite fields, the rank of an $\Sigma\Pi\Sigma$ identity can be as large as $\Omega(k \log d)$ [KS07], [SS09]. Hence, the $O(k^3 \log d)$ bound proved by [SS09] is almost optimal. As a small bonus, we give a slight improvement upon this bound using our approach. This requires Sylvester-Gallai theorems over arbitrary fields, an interesting question in itself. It was shown that $\mathrm{SG}_2(\mathbb{C}, m) \leq 3$ [EPS06], and certain lower bounds for locally decodable codes implied $\mathrm{SG}_2(\mathbb{F}, m) = O(\log m)$. (Concretely, Corollary 2.9 of [DS06] can be used to prove that $\mathrm{SG}_2(\mathbb{F}, m) = O(\log m)$. This is an extension of theorems in [GKST02] that prove this for $\mathbb{F}_2$.) Other than this, nothing was previously known. One of our auxiliary theorems, of independent interest, gives a high-dimensional Sylvester-Gallai bound for all fields. Applying the stronger version of Theorem 4, we get our rank bound.

Theorem 7 (SG_k for all fields): For any field $\mathbb{F}$ and $k, m \in \mathbb{N}^{>1}$, $\mathrm{SG}_k(\mathbb{F}, m) \leq 9k \lg m$. (There is a construction that shows that $\mathrm{SG}_k(\mathbb{F}_p, m) = \Omega(k \cdot \log_p m)$.)

Let C be a $\Sigma\Pi\Sigma(k, d)$ circuit, over an arbitrary field $\mathbb{F}$, that is simple, minimal and zero. Then, $\mathrm{rk}(C) < 3k^2(\lg 2d)$.

B. History

And now, for a brief history of PIT algorithms. The first randomized polynomial time PIT algorithm, which was a black-box algorithm, was given (independently) by Schwartz [Sch80] and Zippel [Zip79]. Randomized algorithms that use less randomness were given by Chen & Kao [CK00], Lewin & Vadhan [LV98], and Agrawal & Biswas [AB03]. Klivans & Spielman [KS01] observed that even for depth-3 circuits for bounded top fanin, deterministic identity testing was open. Progress towards this was first made by the quasi-polynomial time algorithm of Dvir & Shpilka [DS06]. The problem was resolved by a polynomial time algorithm given by Kayal and Saxena [KS07], with a running time exponential in the top fanin. Both these algorithms were non-black-box. As for black-box algorithms, the authors are quite sure that the reader has heard enough history. Identity tests are known only for very special depth-4 circuits [AM07], [Sax08], [SV09], [KMSV09]. Agrawal and Vinay [AV08] showed that an efficient black-box identity test for depth-4 circuits will actually give a quasi-polynomial black-box test, and subexponential lower bounds, for circuits of *all depths* that compute *low degree* polynomials. Thus, understanding depth-3 identities seems to be a natural first step towards the goal of PIT.

II. Proof Outline, Ideas, and Organization

Our proof of the rank bound comprises of several new ideas, both at the conceptual and the technical levels. Instead of giving proofs in this extended abstract, we will only provide the intuition and the overall argument. We recommend the interested reader to see the full version of this paper [SS10]. The full proof of Theorem 4 is extremely technical, requires many definitions, and involve many algebraic arguments. Our attempt is to convey with main ideas without getting into too much formalism or mathematical details. We describe all the major milestones, many of which are interesting in their own right. Indeed, it is the authors' opinion that the reader has little to gain from simply reading the detailed proofs without getting the essence of the ideas.

The intuition portion is divided into three subsections, each dealing with a separate component of the final proof.

Each portion proves an interesting structural theorem. The three notions that are crucially used and developed are: ideal Chinese remaindering, matchings and Sylvester-Gallai rank bounds. Related notions have appeared (in some form) in the works of Kayal & Saxena [KS07], Saxena & Seshadhri [SS09] and Kayal & Saraf [KS09b] respectively, to prove different kinds of results. The first two steps set up the algebraic framework and prove theorems that hold for all fields. The third step is where the Sylvester-Gallai theorems are brought in.

A. Notation and definitions

We will denote the set $\{1, \dots, n\}$ by $[n]$. We fix the base field to be $\mathbb{F}$, so the circuits compute multivariate polynomials in the *polynomial ring* $R := \mathbb{F}[x_1, \dots, x_n]$. We use $\mathbb{F}^*$ to denote $\mathbb{F} \setminus \{0\}$.

A *linear form* is a linear polynomial in R with zero constant term. We will denote the set of all linear forms by $L(R) := \{\sum_{i=1}^n a_i x_i \mid a_1, \dots, a_n \in \mathbb{F}\}$. Clearly, $L(R)$ is a vector (or linear) space over $\mathbb{F}$ and that will be quite useful. Much of what we do shall deal with *multi*-sets of linear forms (sometimes polynomials in R), equivalence classes inside them, and various maps across them. A *list* of linear forms is a multi-set of forms with an arbitrary order associated with them. The actual ordering is unimportant: we will heavily use maps between lists, and the ordering allows us to define these maps unambiguously. The usual set operations between lists can be naturally defined.

Definition 8: We collect some important definitions from [SS09]:

[Multiplication term, $L(\cdot)$ & $M(\cdot)$] A *multiplication term* f is an expression in R given as (the product may have repeated ℓ's), $f := c \cdot \prod_{\ell \in S} \ell$, where $c \in \mathbb{F}^*$ and S is a list of nonzero linear forms. The *list of linear forms in* f, $L(f)$, is just the list S of forms occurring in the product above. For a list S of linear forms we define the *multiplication term of* S, $M(S)$, as $\prod_{\ell \in S} \ell$ or 1 if $S = \phi$.

[Forms in a Circuit] We will represent a $\Sigma\Pi\Sigma(k,d)$ circuit C as a sum of k multiplication terms of degree d, $C = \sum_{i=1}^k T_i$. The list of *linear forms occurring in* C is $L(C) := \bigcup_{i\in[k]} L(T_i)$. Note that $L(C)$ is a list of size exactly kd. The *rank of* C, $\mathrm{rk}(C)$, is just the number of linearly independent linear forms in $L(C)$. (Remark: for the purposes of this paper T_i's are given in circuit representation and thus the list $L(T_i)$ is unambiguously defined from C)

[Similar forms] For any two polynomials $f, g \in R$ we call f *similar to* g if there exists $c \in \mathbb{F}^*$ such that $f = cg$. We say f *is similar to* g *mod* I, for some ideal I of R, if there is some $c \in \mathbb{F}^*$ such that $f \equiv cg (\text{mod } I)$. Note that "similarity mod I" is an equivalence relation (reflexive, symmetric and transitive) and partitions any list of polynomials into equivalence classes.

[Span sp$(\cdot)$] For any $S \subseteq L(R)$ we let $\mathrm{sp}(S) \subseteq L(R)$ be the *linear span* of the linear forms in S over the field $\mathbb{F}$. (Conventionally, $\mathrm{sp}(\emptyset) = \{0\}$.)

[Matchings] Let U, V be lists of linear forms and I be a subspace of $L(R)$. An *I-matching π between U, V* is a bijection π between lists U, V such that: for all $\ell \in U$, $\pi(\ell) \in \mathbb{F}^* \ell + I$.

When f, g are multiplication terms, an *I-matching between* f, g would mean an I-matching between $L(f), L(g)$.

B. Step 1: Matching the Gates in an Identity

We will show that all the multiplication terms of a minimal $\Sigma\Pi\Sigma$ identity can be matched by a low rank space K, spanned by "few" linear forms in $L(R)$.

Theorem 9 (Matching-Nucleus): Let $C = T_1 + \cdots + T_k$ be a $\Sigma\Pi\Sigma(k,d)$ circuit that is minimal and zero. Then there exists a linear subspace K of $L(R)$ such that:

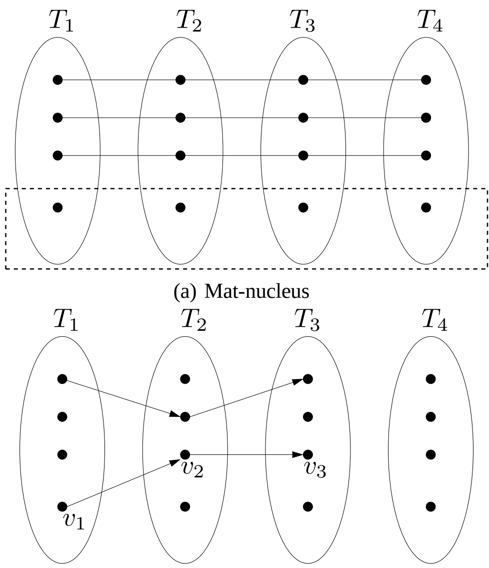
1) $\mathrm{rk}(K) < k^2$.
2) $\forall i \in [k]$, there is a K-matching π_i between T_1, T_i.

The idea of matchings within identities was first introduced in [SS09], but nothing as powerful as this theorem has been proven. This theorem gives us a space of small rank, *independent of* d, that contains most of the "complexity" of C. All forms in C outside K are just mirrored in the various terms. This starts connecting the algebra of depth-3 identities to a combinatorial structure. Indeed, the graphical picture (explained in detail below) that this theorem provides, really gives an intuitive grasp on these identities. The proof of this involves some interesting generalizations of the Chinese Remainder Theorem to some special ideals.

Definition 10 (mat-nucleus): Let C be a minimal $\Sigma\Pi\Sigma(k,d)$ identity. The linear subspace K given by Theorem 9 is called *mat-nucleus of* C.

The notion of mat-nucleus is easier to see in the representation of the $\Sigma\Pi\Sigma(4,d)$ circuit $C = \sum_{i\in[4]} T_i$ given in Figure 1a. The four bubbles refer to the four multiplication terms of C and the points inside the bubbles refer to the linear forms in the terms. The proof of Theorem 9 gives mat-nucleus as the space generated by the linear forms in the dotted box. The linear forms that are not in mat-nucleus lie "above" the mat-nucleus and are all (mat-nucleus)-matched, i.e. $\forall \ell \in (L(T_1) \setminus \text{mat-nucleus})$, there is a form similar to ℓ modulo mat-nucleus in each $(L(T_i) \setminus \text{mat-nucleus})$. Thus the essence of Theorem 9 is: the mat-nucleus part of the terms of C has low rank k^2, while the part of the terms above mat-nucleus all look "similar".

Proof Idea for Theorem 9: The key insight in the construction of mat-nucleus is a reinterpretation of the non-black-box identity test of Kayal & Saxena [KS07] as a structural result for $\Sigma\Pi\Sigma$ identities. Roughly speaking,



(a) Mat-nucleus

(b) Paths

Figure 1

[KS07] showed that $C = 0$ iff for every *path* (v_1, v_2, v_3) (where $v_i \in L(T_i)$): $T_4 \equiv 0 (\text{mod } v_1, v_2, v_3)$ or in ideal terms, $T_4 \in \langle v_1, v_2, v_3 \rangle$. (This is technically *false*, but it portrays the right idea.) Paths are depicted in Figure 1b. Thus, it is enough to go through all the d^3 paths to certify the zeroness of C. This is why the time complexity of the identity test of [KS07] is dominated by d^k. Now if we are given a $\Sigma\Pi\Sigma(4, d)$ identity C which is *minimal*, then we know that $T_1 + T_2 + T_3 \neq 0$. Thus, by applying the above interpretation of [KS07] to $T_1 + T_2 + T_3$ we will get a path (v_1, v_2) such that $T_3 \notin \langle v_1, v_2 \rangle$. Since $C = 0$ this means that $T_3 + T_4 \equiv 0 (\text{mod } v_1, v_2)$ but $T_3, T_4 \not\equiv 0 (\text{mod } v_1, v_2)$ (if T_4 is in $\langle v_1, v_2 \rangle$ then so will be T_3). Thus, $T_3 \equiv -T_4 (\text{mod } v_1, v_2)$ is a nontrivial congruence and it immediately gives us a $\langle v_1, v_2 \rangle$-matching between T_3, T_4. By repeating this argument with a different permutation of the terms we could match different terms (by a different ideal), and finally we expect to match all the terms (by the union of the various ideals).

This argument has numerous technical problems, the most important one being that it does not really work. But all issues can be taken care of by suitable algebraic generalizations. A major stumbling block is the presence of *repeating* forms. It could happen that $(\text{mod } v_1)$, v_2 occurs in many terms, or in the same term with a higher power. The most important tool developed is an ideal version of Chinese remaindering that forces us to consider not just linear forms v_1, v_2, but *multiplication terms* v_1, v_2 dividing T_1, T_2 respectively.

C. Step 2: Certificate for Linear Independence of Gates

Theorem 9 gives us a space K, of rank $< k^2$, that matches T_1 to each term T_i. In particular, this means that the list $L_K(T_i) := L(T_i) \cap K$ has the same cardinality d' for each $i \in [k]$. In fact, if we look at the corresponding multiplication terms $K_i := M(L_K(T_i))$, $i \in [k]$, then they again form a $\Sigma\Pi\Sigma(k, d')$ identity! Precisely, $C' = \sum_{i\in[k]} \alpha_i K_i$ for some α_i's in $\mathbb{F}^*$ is an identity. We would like C' to somehow mimic the structure of C. Of course C' is simple but is it again minimal? Unfortunately, it may not be. As we will see in Step 3, when C' somewhat "mirrors" the structure of C, then bounding the rank of the forms "outside" K becomes possible. Step 2 involves increasing the space K (but not by too much) that gives us a C' with the right behavior. Specifically, if $T_1, \ldots, T_{k'}$ are *linearly independent* (i.e. $\nexists\, \overline{\beta} \in \mathbb{F}^{k'} \setminus \{\overline{0}\}$ s.t. $\sum_{i\in[k']} \beta_i T_i = 0$), then so are $K_1, \ldots, K_{k'}$. The following can be seen as an important structural theorem of depth-3 identities.

Theorem 11 (Nucleus): Let $C = \sum_{i\in[k]} T_i$ be a minimal $\Sigma\Pi\Sigma(k, d)$ identity and let $\{T_i | i \in \mathcal{I}\}$ be a maximal set of linearly independent terms ($1 \leq k' := |\mathcal{I}| < k$). Then there exists a linear subspace K of $L(R)$ such that:

1) $\text{rk}(K) < 2k^2$.
2) $\forall i \in [k]$, there is a K-matching π_i between T_1, T_i.
3) (Define $\forall i \in \mathcal{I}$, $K_i := M(L_K(T_i))$.) The terms $\{K_i | i \in \mathcal{I}\}$ are linearly independent.

Definition 12 (nucleus): Let C be a minimal $\Sigma\Pi\Sigma(k, d)$ identity. The linear subspace K given by Theorem 11 is called the *nucleus of* C. The subspace K induces an identity $C' = \sum_{i\in[k]} \alpha_i K_i$ which we call the *nucleus identity*.

The notion of the nucleus is easier to grasp when C is a $\Sigma\Pi\Sigma(k, d)$ identity that is *strongly minimal*, i.e. $T_1, \ldots, T_{k-1}$ are linearly independent. Clearly, such a C is also minimal For such a C, Theorem 11 gives a nucleus K such that the corresponding nucleus identity is strongly minimal. The structure of C is very strongly represented by C'. As a bonus, we actually end up greatly simplifying the polynomial-time PIT algorithm of Kayal & Saxena [KS07] (although we will not discuss this point in detail in this paper).

Proof Idea for Theorem 11: The first two properties in the theorem statement are already satisfied by mat-nucleus of C. So we incrementally add linear forms to the space mat-nucleus till it satisfies property (3) and becomes the nucleus. The addition of linear forms is guided by the ideal version of Chinese remaindering. For convenience assume T_1, T_2, T_3 to be linearly independent. Then, by homogeneity and equal degree, we have an equivalent ideal statement:

$T_2 \notin \langle T_1 \rangle$ and $T_3 \notin \langle T_1, T_2 \rangle$. Even in this general setting the path analogy (used in the last subsection) works and we essentially get linear forms $v_1 \in L(T_1)$ and $v_2 \in L(T_2)$ such that: $T_2 \notin \langle v_1 \rangle$ and $T_3 \notin \langle v_1, v_2 \rangle$. We now add these forms v_1, v_2 to the space mat-nucleus, and call the new space K. It is expected that the new K_1, K_2, K_3 are now linearly independent.

Not surprisingly, the above argument has numerous technical problems. But it can be made to work by careful applications of the ideal version of Chinese remaindering.

D. Step 3: Invoking Sylvester-Gallai Theorems

As explained in Section I-A, we rephrase the standard Sylvester-Gallai theorems in terms of *Sylvester-Gallai closure* and *rank bounds* (Definition 3). Using some linear algebra and combinatorial tricks, we prove the first ever general Sylvester-Gallai bound for all fields.

Theorem 13 (General Sylvester-Gallai): For any field $\mathbb{F}$ and $k, m \in \mathbb{N}^{>1}$, $\text{SG}_k(\mathbb{F}, m) \leq 9k \lg m$.

The following definition is very helpful in applying Sylvester-Gallai rank bounds to our scenario.

Definition 14 (SG operator): **[$\text{SG}_k(\cdot)$]** Let $k, m \in \mathbb{N}^{>1}$. Suppose a set $S \subseteq \mathbb{F}^n$ has rank greater than $\text{SG}_k(\mathbb{F}, m)$ (where $\#S \leq m$). Then, by definition, S is not SG_k-closed. In this situation we say the *k-dimensional Sylvester-Gallai operator* $\text{SG}_k(S)$ (applied on S) returns a set of k linearly independent vectors V in S whose span has no point in $S \setminus V$.

Let C be a simple and strongly minimal $\Sigma\Pi\Sigma(k, d)$ identity. Theorem 11 gives us a nucleus K, of rank $< 2k^2$, that matches T_1 to each term T_i. As seen in Step 2, if we look at the corresponding multiplication terms $K_i := M(L_K(T_i))$, $i \in [k]$, then they again form a $\Sigma\Pi\Sigma(k, d')$ "nucleus identity" $C' = \sum_{i \in [k]} \alpha_i K_i$, for some α_i's in $\mathbb{F}^*$, which is simple and strongly minimal. Define the *non-nucleus part* of T_i as $L_K^c(T_i) := L(T_i) \setminus K$, for all $i \in [k]$ (c in the exponent annotates "complement", since $L(T_i) = L_K(T_i) \sqcup L_K^c(T_i)$). What can we say about the rank of $L_K^c(T_i)$?

Define the *non-nucleus part of C* as $L_K^c(C) := \bigcup_{i \in [k]} L_K^c(T_i)$. Our goal in Step 3 is to bound $\text{rk}(L_K^c(C) \bmod K)$ by $2k$ when the field is $\mathbb{R}$. This will give us a rank bound of $\text{rk}(K) + \text{rk}(L_K^c(C) \bmod K) < (2k^2 + 2k)$ for simple and strongly minimal $\Sigma\Pi\Sigma(k, d)$ identities over $\mathbb{R}$. The proof is mainly combinatorial, based on higher dimensional Sylvester-Gallai theorems and a property of set partitions, with a sprinkling of algebra.

We apply the SG_k operator not directly on the forms in $L(C)$ but on a suitable truncation of those forms. So we need another definition.

Definition 15 (Non-K rank): Let K be a linear subspace of $L(R)$. Then $L(R)/K$ is again a linear space (the *quotient space*). Let S be a list of forms in $L(R)$. The *non-K rank of S* is defined to be $\text{rk}(S \bmod K)$ (i.e. the rank of S when viewed as a subset of $L(R)/K$).

Let C be a $\Sigma\Pi\Sigma(k, d)$ identity with nucleus K. The non-K rank of the non-nucleus part $L_K^c(T_i)$ is called the *non-nucleus rank of T_i*. Similarly, the non-K rank of the non-nucleus part $L_K^c(C) := \bigcup_{i \in [k]} L_K^c(T_i)$ is called the *non-nucleus rank of C*.

We give an example to explain the non-K rank. Let $R = \mathbb{F}[z_1, \cdots, z_n, y_1, \cdots, y_m]$. Suppose $K = \text{sp}(z_1, \cdots, z_n)$ and $S \subset L(R)$. We can take any element ℓ in S and simply drop all the z_i terms, i.e. 'truncate' the z-part of ℓ. This gives a set of linear forms over the y variables. The rank of these is the non-K rank of S.

We are now ready to state the theorem that is proved in Step 3. It basically shows a neat relationship between the non-nucleus part and Sylvester-Gallai.

Theorem 16 (Bound for simple, strongly minimal identities): Let $|\mathbb{F}| > d$. The non-nucleus rank of a simple and strongly minimal $\Sigma\Pi\Sigma(k, d)$ identity over $\mathbb{F}$ is at most $\text{SG}_{k-1}(\mathbb{F}, d)$. More specifically, (for nucleus K) the vectors in $L(C) \setminus K$ form an SG_{k-1}-closed set.

Observe that this theorem together with Theorem 11 gives a complete structure theorem for strongly minimal depth-3 identities. One can make suitable claims for identities that are not strictly minimal. Essentially, we just take a subset of linearly independent terms, say $T_1, \ldots, T_{k'}$, that form a basis for $\{T_i | i \in [k]\}$. We can now construct strongly minimal identities using these terms and apply the above theorem. Specifically, we get the following.

Definition 17 (Independent-fanin): Let $C = \sum_{i \in [k]} T_i$ be a $\Sigma\Pi\Sigma(k, d)$ circuit. The *independent-fanin* of C, ind-fanin(C), is defined to be the size of the maximal $\mathcal{I} \subseteq [k]$ such that $\{T_i | i \in \mathcal{I}\}$ are linearly independent polynomials.

We now state the following stronger version of the main theorem.

Theorem 18 (Final bound): Let $|\mathbb{F}| > d$. The rank of a simple, minimal $\Sigma\Pi\Sigma(k, d)$, independent-fanin k', identity is at most $2k^2 + (k - k') \cdot \text{SG}_{k'}(\mathbb{F}, d)$.

Remark: In particular, the rank of a simple, minimal $\Sigma\Pi\Sigma(k, d)$ identity over reals is at most $2k^2 + (k - k') \cdot \text{SG}_{k'}(\mathbb{R}, d) \leq 2k^2 + (k - k')2(k' - 1) < 3k^2$, proving the main theorem over reals. Likewise, for any $\mathbb{F}$, we get the rank bound of $2k^2 + (k - k') \cdot \text{SG}_{k'}(\mathbb{F}, d)$

$\leq\ 2k^2 + (k-k')9k'\lg d \ \leq\ 2k^2 + \frac{9k^2}{4}\lg d\ <\ 3k^2\lg 2d$, proving the main theorem.

Proof Idea for Theorem 16: Basically, we apply the $\mathrm{SG}_k(\cdot)$ operator on the non-nucleus part of the term T_1, i.e. we treat a linear form $\sum_i a_i x_i$ as the point $(1, \frac{a_2}{a_1}, \ldots, \frac{a_n}{a_1}) \in \mathbb{F}^n$ for the purposes of Sylvester-Gallai and then we consider $\mathrm{SG}_k(L_K^c(T_1))$ assuming that the non-nucleus rank of T_1 is more than $\mathrm{SG}_k(\mathbb{F}, d)$. This application of Sylvester-Gallai is much more direct compared to the methods used in [KS09b]. There, they effectively needed to prove versions of Sylvester-Gallai that dealt with colored points and needed a *hyperplane decomposition* property after applying a $\mathrm{SG}_{k^{O(k)}}(\cdot)$ operator on $L(C)$. Since, modulo the nucleus, all multiplication terms look essentially the same, it suffices to focus attention on just one of them. Hence, we apply the SG_k-operator on a single multiplication term.

Assume C is a simple, strongly minimal $\Sigma\Pi\Sigma(k,d)$ identity with terms $\{T_i | i \in [k]\}$ and let K be its nucleus given by Step 2. It will be convenient for us to fix a linear form $y_0 \in L(R)^*$ and a subspace U of $L(R)$ such that we have the following *orthogonal* vector space decomposition $L(R) = \mathbb{F}y_0 \oplus U \oplus K$. This means for any form $\ell \in L(R)$, there is a unique way to express $\ell = \alpha y_0 + u + v$, where $\alpha \in \mathbb{F}$, $u \in U$ and $v \in K$. Furthermore, we will assume wlog that for every form $\ell \in L_K^c(T_1)$ the corresponding α is nonzero, i.e. each form in $L_K^c(T_1)$ is *monic* wrt y_0. Technically, we do not need the extra variable y_0 and can work in a projective space. Nonetheless, it makes the presentation easier.

Definition 19 (trun($\cdot$)): Fix a decomposition $L(R) = \mathbb{F}y_0 \oplus U \oplus K$. For any form $\ell \in L_K^c(T_1)$, there is a unique way to express $\ell = \alpha y_0 + u + v$, where $\alpha \in \mathbb{F}^*$, $u \in U$ and $v \in K$.

The *truncated form trun*(ℓ) is the linear form obtained by dropping the K part and normalizing, i.e. $\mathrm{trun}(\ell) := y_0 + \alpha^{-1}u$.

Given a list of forms S we define $\mathrm{trun}(S)$ to be the corresponding *set* (thus no repetitions) of truncated forms.

To be precise, we fix a basis $\{y_1, \ldots, y_{\mathrm{rk}(U)}\}$ of U so that each form in $\mathrm{trun}(L_K^c(T_1))$ has representation $y_0 + \sum_{i\geq 1} a_i y_i$ (a_i's $\in \mathbb{F}$). We view each such form as the *point* $(1, a_1, \ldots, a_{\mathrm{rk}(U)})$ while applying Sylvester-Gallai on $\mathrm{trun}(L_K^c(T_1))$. Assume, for the sake of contradiction, that the non-nucleus rank of T_1, $\mathrm{rk}(\mathrm{trun}(L_K^c(T_1))) > \mathrm{SG}_{k-1}(\mathbb{F}, d)$. Therefore, $\mathrm{SG}_{k-1}(\mathrm{trun}(L_K^c(T_1)))$ gives $(k-1)$ linearly independent forms $\ell_1, \ldots, \ell_{k-1} \in (y_0 + U)$ whose span contains no *other* linear form of $\mathrm{trun}(L_K^c(T_1))$.

For simplicity of exposition, let us fix $k = 4$, K spanned by z's, U spanned by y's and $\ell_i = y_0 + y_i$ ($i \in [3]$). Note that (by definition) $\mathrm{trun}(\alpha y_0 + \sum_i \alpha_i z_i + \sum_i \beta_i y_i) = y_0 + \sum_i \frac{\beta_i}{\alpha} y_i$. We want to derive a contradiction using the SG_3-operator output $(y_0+y_1, y_0+y_2, y_0+y_3)$ and the fact that C is a simple, strongly minimal $\Sigma\Pi\Sigma(4,d)$ identity. Consider the setting given in Figure 2. Suppose the linear forms in C that are similar to a form in $\{y_0 + y_i + K | i \in [3]\}$ are exactly those depicted in the figure. All forms within a row are K-matched. We would like to find forms $\ell_1', \ell_2', \ell_3'$ with the following properties: (1) $\ell_i' \equiv c_i \ell_i (\mathrm{mod}\ K)$ (for some constant c_i). (2) There exists some j such that no ℓ_i' divides T_j but for each T_l ($l \neq j$), some ℓ_i' divides T_l. In this situation, we can choose $\ell_1' = y_0+y_1+z_1$, $\ell_2' = y_0+y_2+z_2$, and $\ell_3' = -y_0 - y_3 + z_2$. None of these divides T_4. Observe that the triple $(y_0+y_1+z_1, y_0+y_2+z_2, y_0+y_3+z_1)$ does not satisfy these conditions, since no appropriate T_j can be found.

Take C modulo the ideal $I := \langle y_0+y_1+z_1, y_0+y_2+z_2, -y_0-y_3+z_2\rangle$. It is easy to see that $C \equiv T_4 (\mathrm{mod}\ I)$, so I "kills" the first three terms. Since C is an identity, $T_4 \in I$. Thus, there is a form $\ell \in L(T_4)$ such that $\ell \in \mathrm{sp}(\ell_1', \ell_2', \ell_3')$. Since no form from ℓ_i' divides T_4, so ℓ must be a non-trivial combination of these forms. By the matching property, there exists some form $\hat{\ell} \in L(T_1)$ such that $\mathrm{trun}(\ell) = \mathrm{trun}(\hat{\ell})$. In other words, $\mathrm{trun}(\ell) \in \mathrm{trun}(L_K^c(T_1))$. But that contradicts the fact that $(\ell_1', \ell_2', \ell_3')$ form an SG_3-tuple. This implies that the non-nucleus rank of C is at most $\mathrm{SG}_3(\mathbb{F}, d)$.

The approach above worked because we were lucky enough to find $\ell_1', \ell_2', \ell_3'$ with the right properties. Can we always do this? No, because of repeating forms. Suppose, after going modulo form ℓ, the circuit looks like $x^3y + 2x^2y^2 + xy^3 = 0$. This is not simple, but *it does not have to be*. We are only guaranteed that the original circuit is simple. Once we go modulo ℓ, that property is lost. Now, the choice of *any* form kills all terms. We will use our more powerful Chinese remaindering tools and the nucleus properties to deal with this. The minimality of the nucleus identity plays a crucial role here and helps us deal with such situations. We have to prove a special theorem about partitions of $[k]$ and use strong minimality (which we did not use in the above sketch).

III. Conclusion

In this work we developed the strongest methods, to date, to study depth-3 identities. The ideal methods hinge on a classification of zerodivisors of the ideals generated by gates of a $\Sigma\Pi\Sigma$ circuit. That is useful in proving an ideal version of Chinese remaindering tailor-made for $\Sigma\Pi\Sigma$ circuits, which is in turn useful to show a connection between all the gates involved in an identity. As a byproduct, it shows the existence of a low rank *nucleus identity* C' inside *any* given $\Sigma\Pi\Sigma(k,d)$ identity C (when C is not minimal, C' can still be defined but it might not be homogeneous). The properties of the nucleus identity are an important part of an identity and it might be useful for PIT to understand (or classify) it further. Can the rank bound for the nucleus

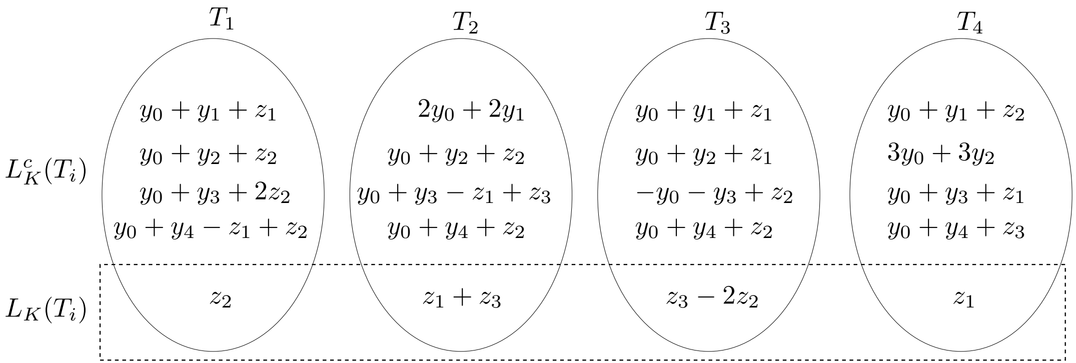


Figure 2

identity be improved to $O(k)$? More importantly, can the rank bound for simple minimal real $\Sigma\Pi\Sigma(k,d)$ identities be improved to $O(k)$? The best constructions known, since [DS06], have rank $4(k-2)$. Over other fields, our upper bound of $O(k^2 \log d)$ still leaves some gap in understanding the exact dependence on k. Of course, the most important question is whether our techniques can help construct a truly polynomial time deterministic (even non-black-box) algorithm for PIT.

We generalize the notion of Sylvester-Gallai configurations to *any* field and define a parameter $\mathrm{SG}_k(\mathbb{F},m)$ associated with field $\mathbb{F}$. This number seems to be a fundamental property of a field, and as we show, is very closely related to $\Sigma\Pi\Sigma$ identities. It would be interesting to obtain bounds for $\mathrm{SG}_k(\mathbb{F},m)$ for different $\mathbb{F}$. For example, as also asked by [KS09b], can we nontrivially bound the number $\mathrm{SG}_k(\mathbb{F},m)$ for interesting fields: $\mathbb{C}$, finite fields with large characteristic, or even p-adic fields? The only known SG_k rank bounds are those for $\mathbb{R}$, $\mathrm{SG}_2(\mathbb{C},m) \le 3$, and $\mathrm{SG}_2(\mathbb{F},m) \le O(\log m)$. We shed (a little) light on SG rank bounds by showing $\mathrm{SG}_k(\mathbb{F},m) = O(k \log m)$. We conjecture: $\mathrm{SG}_k(\mathbb{F},m)$ is $O(k)$ for zero characteristic fields, while $O(k + k \cdot \log_p m)$ for fields of characteristic $p > 1$. The latter would mean that when the characteristic is large ($p \geqslant m$), $\mathrm{SG}_k(\mathbb{F},m) = O(k)$, matching the bounds for zero characteristic fields.

Acknowledgements

We are grateful to Hausdorff Center for Mathematics, Bonn for the kind support, especially, hosting the second author when part of the work was done. The first author thanks Nils Frohberg for several detailed discussions that clarified the topic of incidence geometry and Sylvester-Gallai theorems. The second author is extremely grateful to Ken Clarkson and especially to T. S. Jayram for their suggestions in improving the presentation. We also thank Malte Beecken, Johannes Mittmann (for the high rank SG_k construction over $\mathbb{F}_p$) and Thomas Thierauf for several interesting discussions.

References

[AB03] M. Agrawal and S. Biswas. Primality and identity testing via Chinese remaindering. *Journal of the ACM*, 50(4):429–443, 2003. (Conference version in FOCS 1999).

[Agr05] M. Agrawal. Proving lower bounds via pseudo-random generators. In *Proceedings of the 25th Annual Foundations of Software Technology and Theoretical Computer Science (FSTTCS)*, pages 92–105, 2005.

[Agr06] M. Agrawal. Determinant versus permanent. In *Proceedings of the 25th International Congress of Mathematicians (ICM)*, volume 3, pages 985–997, 2006.

[AM07] V. Arvind and P. Mukhopadhyay. The monomial ideal membership problem and polynomial identity testing. In *Proceedings of the 18th International Symposium on Algorithms and Computation (ISAAC)*, pages 800–811, 2007.

[AS09] M. Agrawal and R. Saptharishi. Classifying polynomials and identity testing. Technical report, IIT Kanpur, http://www.cse.iitk.ac.in/manindra/survey/Identity.pdf, 2009.

[AV08] M. Agrawal and V. Vinay. Arithmetic circuits: A chasm at depth four. In *Proceedings of the 49th Annual Symposium on Foundations of Computer Science (FOCS)*, pages 67–75, 2008.

[BE67] W. Bonnice and M. Edelstein. Flats associated with finite sets in P^d. *Niew. Arch. Wisk.*, 15:11–14, 1967.

[BM90] P. Borwein and W. O. J. Moser. A survey of Sylvesters problem and its generalizations. *Aequationes Mathematicae*, 40(1):111–135, 1990.

[CK00] Z. Chen and M. Kao. Reducing randomness via irrational numbers. *SIAM J. on Computing*, 29(4):1247–1256, 2000. (Conference version in STOC 1997).

[DS06] Z. Dvir and A. Shpilka. Locally decodable codes with 2 queries and polynomial identity testing for depth 3 circuits. *SIAM J. on Computing*, 36(5):1404–1434, 2006. (Conference version in STOC 2005).

[EPS06] N. D. Elkies, L. M. Pretorius, and C. J. Swanepoel. Sylvester-Gallai theorems for complex numbers and quaternions. *Discrete & Computational Geometry*, 35(3):361–373, 2006.

[GKST02] O. Goldreich, H. Karloff, L. Schulman, and L. Trevisan. Lower bounds for linear locally decodable codes and private information retrieval. In *Proceedings of the 17th Annual Computational Complexity Conference (CCC)*, pages 175–183, 2002.

[Han65] S. Hansen. A generalization of a theorem of Sylvester on the lines determined by a finite point set. *Mathematica Scandinavia*, 16:175–180, 1965.

[HS80] J. Heintz and C. P. Schnorr. Testing polynomials which are easy to compute (extended abstract). In *Proceedings of the twelfth annual ACM Symposium on Theory of Computing*, pages 262–272, New York, NY, USA, 1980.

[KI04] V. Kabanets and R. Impagliazzo. Derandomizing polynomial identity tests means proving circuit lower bounds. *Computational Complexity*, 13(1):1–46, 2004. (Conference version in STOC 2003).

[KMSV09] Z. Karnin, P. Mukhopadhyay, A. Shpilka, and I. Volkovich. Deterministic identity testing of depth 4 multilinear circuits with bounded top fan-in. Technical Report TR09-116, ECCC, http://eccc.hpi-web.de/report/2009/116/, 2009.

[KS01] A. Klivans and D. A. Spielman. Randomness efficient identity testing of multivariate polynomials. In *Proceedings of the 33rd Symposium on Theory of Computing (STOC)*, pages 216–223, 2001.

[KS07] N. Kayal and N. Saxena. Polynomial identity testing for depth 3 circuits. *Computational Complexity*, 16(2):115–138, 2007. (Conference version in CCC 2006).

[KS08] Z. Karnin and A. Shpilka. Deterministic black box polynomial identity testing of depth-3 arithmetic circuits with bounded top fan-in. In *Proceedings of the 23rd Annual Conference on Computational Complexity (CCC)*, pages 280–291, 2008.

[KS09a] Z. S. Karnin and A. Shpilka. Reconstruction of generalized depth-3 arithmetic circuits with bounded top fan-in. In *Proceedings of the 24th Annual Conference on Computational Complexity (CCC)*, pages 274–285, 2009.

[KS09b] N. Kayal and S. Saraf. Blackbox polynomial identity testing for depth 3 circuits. In *Proceedings of the 50th Annual Symposium on Foundations of Computer Science (FOCS)*, 2009.

[LV98] D. Lewin and S. Vadhan. Checking polynomial identities over any field: Towards a derandomization? In *Proceedings of the 30th Annual Symposium on the Theory of Computing (STOC)*, pages 428–437, 1998.

[Sax08] N. Saxena. Diagonal circuit identity testing and lower bounds. In *Proceedings of the 35th Annual International Colloquium on Automata, Languages and Programming (ICALP)*, pages 60–71, 2008.

[Sax09] N. Saxena. Progress on polynomial identity testing. *Bulletin of the European Association for Theoretical Computer Science (EATCS)- Computational Complexity Column*, (99):49–79, 2009.

[Sch80] J. T. Schwartz. Fast probabilistic algorithms for verification of polynomial identities. *Journal of the ACM*, 27(4):701–717, 1980.

[Shp09] A. Shpilka. Interpolation of depth-3 arithmetic circuits with two multiplication gates. *SIAM J. Comput.*, 38(6):2130–2161, 2009. (Conference version in STOC 2007).

[SS09] N. Saxena and C. Seshadhri. An almost optimal rank bound for depth-3 identities. In *Proceedings of the 24th Annual Conference on Computational Complexity (CCC)*, pages 137–148, 2009.

[SS10] N. Saxena and C. Seshadhri. From Sylvester-Gallai configurations to rank bounds: Improved black-box identity test for depth-3 circuits. Technical Report TR10-013, ECCC, http://eccc.hpi-web.de/report/2010/013/, 2010.

[SV09] A. Shpilka and I. Volkovich. Improved polynomial identity testing for read-once formulas. In *Proceedings of the 13th International Workshop on Randomization and Computation (RANDOM)*, pages 700–713, 2009.

[Zip79] R. Zippel. Probabilistic algorithms for sparse polynomials. In *Proceedings of the International Symposium on Symbolic and Algebraic Manipulation (EUROSAM)*, pages 216–226, 1979.