DETERMINISTIC IDENTITY TESTING PARADIGMS FOR BOUNDED TOP-FANIN DEPTH-4 CIRCUITS*

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Abstract. Polynomial Identity Testing (PIT) is a fundamental computational problem. The famous depth-4 reduction result by Agrawal and Vinay (FOCS 2008) has made PIT for depth-4 circuits an enticing pursuit. A restricted depth-4 circuit computing a n-variate degree-d polynomial of the form $\sum_{i=1}^{k} \prod_j g_{ij}$, where $\deg g_{ij} \leq \delta$ is called $\Sigma^{[k]}\Pi\Sigma\Pi^{[\delta]}$ circuit. On further restricting g_{ij} to be sum of univariates we obtain $\Sigma^{[k]}\Pi\Sigma\wedge$ circuits. The largely open, special-cases of $\Sigma^{[k]}\Pi\Sigma\Pi^{[\delta]}$ for constant k and δ, and $\Sigma^{[k]}\Pi\Sigma\wedge$ have been a source of many great ideas in the last two decades. For eg. depth-3 ideas of Dvir and Shpilka (STOC 2005), Kayal and Saxena (CCC 2006), and Saxena and Seshadhri (FOCS 2010 and STOC 2011). Further, depth-4 ideas of Beecken, Mittmann and Saxena (ICALP 2011), Saha, Saxena and Saptharishi (Comput.Compl. 2013), Forbes (FOCS 2015), and Kumar and Saraf (CCC 2016). Additionally, geometric Sylvester-Gallai ideas of Kayal and Saraf (FOCS 2009), Shpilka (STOC 2019), and Peleg and Shpilka (CCC 2020, STOC 2021). Very recently, a subexponential-time *blackbox* PIT algorithm for constant-depth circuits was obtained via lower bound breakthrough of Limaye, Srinivasan, Tavenas (FOCS 2021). We solve two of the basic underlying open problems in this work.

We give the *first* polynomial-time PIT for $\Sigma^{[k]}\Pi\Sigma\wedge$. We also give the *first* quasipolynomial time *blackbox* PIT for both $\Sigma^{[k]}\Pi\Sigma\wedge$ and $\Sigma^{[k]}\Pi\Sigma\Pi^{[\delta]}$. A key technical ingredient in all the three algorithms is how the *logarithmic derivative*, and its power-series, modify the top Π-gate to $\wedge$.

Key words. Polynomial identity testing, hitting set, depth-4 circuits

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1. Introduction: PIT & beyond.

Algebraic circuits are natural algebraic analog of boolean circuits, with the logical operations being replaced by $+$ and $\times$ operations over the underlying field. The study of algebraic circuits comprise the large study of algebraic complexity, mainly pioneered (and formalized) by Valiant [93]. A central problem in algebraic complexity is an algorithmic design problem, known as Polynomial Identity Testing (PIT): given an algebraic circuit $\mathcal{C}$ over a field $\mathbb{F}$ and input variables $x_1, \ldots, x_n$, determine whether $\mathcal{C}$ computes the identically zero polynomial. PIT has found numerous applications and connections to other algorithmic problems. Among the examples are algorithms for finding perfect matchings in graphs [63, 67, 27], primality testing [4], polynomial factoring [56, 22], polynomial equivalence [24], reconstruction algorithms [52, 89, 48] and the existence of algebraic natural proofs [16, 57]. Moreover, efficient design of PIT algorithms is intrinsically connected to proving strong lower bounds [43, 1, 46, 26, 33, 17, 23]. Interestingly, PIT also emerges in many fundamental results in complexity theory such as $\mathsf{IP} = \mathsf{PSPACE}$ [88, 64], the PCP theorem [10, 11], and the overarching Geometric Complexity Theory (GCT) program towards $\mathsf{P} \neq \mathsf{NP}$ [69, 68, 36, 45].

There are broadly two settings in which the PIT question can be framed. In the *whitebox* setup, we are allowed to look inside the wirings of the circuit, while in the *blackbox* setting we can only evaluate the circuit at some points from the given

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domain. There is a very simple randomized algorithm for this problem - evaluate the polynomial at a random point from a large enough domain. With very high probability, a nonzero polynomial will have a nonzero evaluation; this is famously known as the Polynomial Identity Lemma [71, 18, 95, 87]. It has been a long standing open question to derandomize this algorithm.

For many years, blackbox identity tests were only known for depth-2 circuits which compute sparse polynomials [13, 53]. In a surprising result, Agrawal and Vinay [7] showed that a complete derandomization of blackbox identity testing for just depth-4 algebraic circuits ($\Sigma\Pi\Sigma\Pi$) already implies a near complete derandomization for the general PIT problem. More recent depth reduction results [54, 40], and the bootstrapping phenomenon [2, 58, 38, 9] show that even PIT for very restricted classes of depth-4 circuits (*even* depth-3) would have very interesting consequences for PIT of general circuits. These results make the identity testing regime for depth-4 circuits, a very meaningful pursuit.

Three PITs in one-shot. Following the same spirit, here we solve three important (and open) PIT questions. We give the first deterministic polynomial-time whitebox PIT algorithm for the bounded sum of product of sum of univariates circuits [76, Open Prob. 2]. Further, we give a quasipolynomial-time blackbox algorithm for the same class of circuits. These circuits are denoted by $\Sigma^{[k]}\Pi\Sigma\wedge$ and compute polynomials of the form $\Sigma_{i\in[k]}\Pi_j\,(g_{ij1}(x_1)+\cdots+g_{ijn}(x_n))$.

> *Whitebox and Blackbox PIT for the $\Sigma^{[k]}\Pi\Sigma\wedge$ circuits is in polynomial and quasi-polynomial time respectively.*

A similar technique also gives a quasi-polynomial time blackbox PIT algorithm for the bounded sum of product of bounded degree sparse polynomials circuits. They are denoted by $\Sigma^{[k]}\Pi\Sigma\Pi^{[\delta]}$ (where k and δ are constants).

> *Blackbox PIT for the $\Sigma^{[k]}\Pi\Sigma\Pi^{[\delta]}$ circuits is in quasi-polynomial time.*

$\Sigma^{[k]}\Pi\Sigma\Pi^{[\delta]}$ circuits compute polynomials which are of the form $\Sigma_{i\in[k]}\Pi_j g_{ij}(\boldsymbol{x})$, where $\mathsf{deg}(g_{ij})\le\delta$. Even $\delta=2$ was a challenging open problem [59, Open Problem 2].

1.1. Main results: An analytic detour to three PITs. Though some attempts have been made to solve PIT for $\Sigma^{[k]}\Pi\Sigma\wedge$, an efficient PIT for $k\ge 3$ *even* in the whitebox settings remains open, see [76, Open Prob. 2]. Our first result addresses this problem and designs a polynomial time algorithm (Algorithm 3.1). In our pursuit we discover an analytic and non-ideal based new technique which we refer as DiDI. Throughout the paper, we will work with $\mathbb{F}=\mathbb{Q}$, though all the results hold for field of large characteristic.

THEOREM 1.1 (Whitebox $\Sigma^{[k]}\Pi\Sigma\wedge$ PIT). *There is a deterministic, whitebox $s^{O(k\,7^k)}$-time PIT algorithm for $\Sigma^{[k]}\Pi\Sigma\wedge$ circuits of size s, over $\mathbb{F}[\boldsymbol{x}]$.*

Remark 1.2.
1. Case $k\le 2$ can be solved by invoking [76, Theorem 5.2]; but $k\ge 3$ was open.
2. Our technique *necessarily* blows up the exponent exponentially in k. In particular, it would be interesting to design an efficient time algorithm when $k=\Theta(\log s)$.
3. It is not clear if the current technique gives PIT for $\Sigma^{[k]}\Pi\Sigma\wedge^{[2]}$ circuits, i.e. sum of *bi*variate polynomials computed and fed into the top product gate.

Next, we go to the blackbox setting and address two models of interest, namely—$\Sigma^{[k]}\Pi\Sigma\wedge$ and $\Sigma^{[k]}\Pi\Sigma\Pi^{[\delta]}$, where k,δ are constants. Our work builds on previous ideas for unbounded top fanin (1) Jacobian [5], (2) the known blackbox PIT for $\Sigma\wedge\Sigma\wedge$ and $\Sigma\wedge\Sigma\Pi^{[\delta]}$ [41, 29] while maneuvering with an analytic approach *via* power-series, which unexpectedly *reduces* the top Π-gate to a ∧-gate.

THEOREM 1.3 (Blackbox depth-4 PIT).

1. *There is a* $s^{O(k \log\log s)}$ *time blackbox PIT algorithm for* $\Sigma^{[k]}\Pi\Sigma\wedge$ *circuits of size* s*, over* $\mathbb{F}[\boldsymbol{x}]$.
2. *There is a* $s^{O(\delta^2\, k\, \log s)}$ *time blackbox PIT algorithm for* $\Sigma^{[k]}\Pi\Sigma\Pi^{[\delta]}$ *circuits of size* s*, over* $\mathbb{F}[\boldsymbol{x}]$.

Remark 1.4.

1. Theorem 1.3 (b) has a *better* dependence on k, but *worse* on s, than Theorem 1.1. Our results are quasipoly-time even up to $k, \delta = \mathsf{poly}(\log s)$.
2. Theorem 1.3 (a) is better than Theorem 1.3 (b), because $\Sigma\wedge\Sigma\wedge$ has a faster algorithm than $\Sigma\wedge\Sigma\Pi^{[\delta]}$.
3. Even for $\Sigma^{[3]}\Pi\Sigma\wedge$ and $\Sigma^{[3]}\Pi\Sigma\Pi^{[3]}$ models, we leave the *poly*-time blackbox question open.

1.2. Prior works on related models. In the last two decades, there has been a surge of results on identity testing for restricted classes of bounded depth algebraic circuits (e.g. 'locally' bounded independence, bounded read/occur, bounded variables). There have been numerous results on PIT for depth-3 circuits with bounded top fanin (known as $\Sigma^{[k]}\Pi\Sigma$-circuits). Divir and Shpilka [25] gave the first quasipolynomial-time deterministic whitebox algorithm for $k = O(1)$, using rank based methods, which finally lead Karnin and Shpilka [49] to design algorithm of same complexity in the blackbox setting. Kayal and Saxena [51] gave the first polynomial-time algorithm of the same. Later, a series of works in [84, 85, 86, 5] generalized the model and gave $n^{O(k)}$-time algorithm when the algebraic rank of the product polynomials are bounded.

There has also been some progress on PIT for restricted classes of depth-4 circuits. A quasipolynomial-time blackbox PIT algorithm for *multilinear* $\Sigma^{[k]}\Pi\Sigma\Pi$-circuits was designed in [47], which was further improved to a $n^{O(k^2)}$-time deterministic algorithm in [80]. A quasipolynomial blackbox PIT was given in [12, 59] when algebraic rank of the irreducible factors in each multiplication gate as well as the bottom fanin are bounded. Further interesting restrictions like sum of product of fewer variables, and more structural restrictions have been exploited, see [32, 6, 29, 66, 60]. Some progress has also been made for bounded top-fanin and bottom-fanin depth-4 circuits via incidence geometry [39, 90, 73]. In fact, very recently, [74] gave a polynomial-time blackbox PIT for $\Sigma^{[3]}\Pi\Sigma\Pi^{[2]}$-circuits.

The authors recently generalised their novel DiDI-technique to solve 'border PIT' of depth-4 circuits [20]. Specifically, they give a $s^{O(k\cdot 7^k\cdot \log logs)}$ time and $s^{O(\delta^2\cdot k\cdot 7^k\cdot \log s)}$ time blackbox PIT algorithm for $\overline{\Sigma^{[k]}\Pi\Sigma\wedge}$ and $\overline{\Sigma^{[k]}\Pi\Sigma\Pi^{[\delta]}}$ respectively. By definition, border classes capture exact complexity classes, hence border PIT results seeminly subsumes the results we present in this paper. However, the whitebox PIT algorithm here is much more efficient than their quasi-poly time blackbox algorithm. Further, the time complexity of blackbox PIT algorithms has a better dependence on k and δ compared to their exponential dependence. Lastly, the proofs in this paper are simpler as we don't have to deal with an infinitesimally close approximation of polynomials in border complexity classes. Very recently, Dutta and Saxena [21] showed an exponential-gap fanin-hierarchy theorem for bounded depth-3 circuits which is also based on a *finer* generalization of the DiDI-technique.

In a breakthrought result by Limaye, Srinivasan and Tavenas [62] the *first* superpolynomial lower bound for constant depth circuits was obtained. Their lower bound

Model	Time	Ref.
$\Sigma^{[k]}\Pi^{[d]}\Sigma$	$\mathsf{poly}(n, d^k)$	[85]
Multilinear $\Sigma^{[k]}\Pi\Sigma\Pi$	$\mathsf{poly}(n^{O(k^2)})$	[80, 5]
$\Sigma\Pi\Sigma\Pi$ of bounded trdeg	$\mathsf{poly}(s^{\mathsf{trdeg}})$	[12]
$\Sigma^{(k)}\Pi\Sigma\Pi^{[d]}$ of bounded *local* trdeg	$\mathsf{QP}(n)$	[60]
$\Sigma^{[3]}\Pi\Sigma\Pi^{[2]}$	$\mathsf{poly}(n, d)$	[74]
$\overline{\Sigma^{[k]}\Pi\Sigma\wedge}$	$s^{O(k \cdot 7^k \cdot \log\log s)}$	[20]
$\overline{\Sigma^{[k]}\Pi\Sigma\Pi^{[\delta]}}$	$s^{O(\delta^2 \cdot k \cdot 7^k \cdot \log s)}$	[20]
$\Sigma\Pi\Sigma\Pi$	$\mathrm{SUBEXP}(n)$	[62]
Whitebox $\Sigma^{[k]}\Pi\Sigma\wedge$	$s^{O(k\, 7^k)}$	This work.
$\Sigma^{[k]}\Pi\Sigma\wedge$	$s^{O(k \log\log s)}$	This work.
$\Sigma^{[k]}\Pi\Sigma\Pi^{[\delta]}$	$s^{O(\delta^2\, k\, \log s)}$	This work.

TABLE 1
Time complexity comparison of PIT algorithms related to $\Sigma\Pi\Sigma\Pi$ circuits

result, together with the 'hardness vs randomness' tradeoff result of [17] gives the *first* deterministic blackbox PIT algorithm for general depth-4 circuits which runs in $s^{O(n^{\epsilon})}$ for all real $\epsilon > 0$. Their result is the first *sub*exponential time PIT algorithm for depth-4 circuits. Moreover, compared to their algorithm, our quasipoly time blackbox and polynomial time whitebox algorithms are significantly faster.

Limitations of known techniques. People have studied depth-4 PIT only with extra restrictions, mostly due to the limited applicability of the existing techniques as they were tailor-made for the specific models and do not generalize. E.g. the previous methods handle $\delta = 1$ (i.e. linear polynomials at the bottom) or $k = 2$ (via *factoring*, [76]). While $k = 2$ to 3, or $\delta = 1$ to 2 (i.e. 'linear' to 'quadratic') already demands a qualitatively different approach.

Whitebox $\Sigma^{[k]}\Pi\Sigma\wedge$ model generalizes the famous bounded top fanin depth-3 circuits $\Sigma^{[k]}\Pi\Sigma$ of [51]; but their Chinese Remaindering (CR) method, loses applicability and thus fails to solve even a slightly more general model. The blackbox setting involved similar 'certifying path' ideas in [85] which could be thought of as general CR. It comes up with an ideal I such that $f \neq 0 \bmod I$ and finally preserves it under a constant-variate linear map. The preservation gets harder (for both $\Sigma^{[k]}\Pi\Sigma\wedge$ and $\Sigma^{[k]}\Pi\Sigma\Pi^{[\delta]}$) due to the increased non-linearity of the ideal I generators. Intuitively, larger δ via ideal-based routes, brings us to the Gröbner basis method (which is doubly-exponential-time in n) [94]. We know that ideals even with 3-generators (analogously $k = 4$) already capture the whole ideal-membership problem [79].

The algebraic-geometric approach to tackle $\Sigma^{[k]}\Pi\Sigma\Pi^{[\delta]}$ has been explored in [12, 39, 66, 37]. The families which satisfy a certain Sylvester–Gallai configuration (called SG-circuits) is the harder case which is conjectured to have constant transcendence degree [39, Conj. 1]. Non-SG circuits is the case where the nonzeroness-certifying-path question reduces to radical-ideal non-membership questions [35]. This is really a variety question where one could use algebraic-geometry tools to design a poly-time blackbox PIT. In fact, very recently, Guo [37] gave a s^{δ^k}-time PIT by constructing explicit variety evasive subspace families. Unfortunately, this is not the case in the ideal non-membership; this scenario makes it much harder to solve $\Sigma^{[k]}\Pi\Sigma\Pi^{[\delta]}$. From this viewpoint, radical-ideal-membership explains well why the intuitive $\Sigma^{[k]}\Pi\Sigma$ methods do not extend to $\Sigma^{[k]}\Pi\Sigma\Pi^{[\delta]}$.

Interestingly, Forbes [29] found a quasipolynomial-time PIT for $\Sigma\wedge\Sigma\Pi^{[\delta]}$ using shifted-partial derivative techniques; but it naively fails when one replaces the $\wedge$-gate by Π (because the 'measure' becomes too large). The duality trick of [81] completely solves whitebox PIT for $\Sigma\wedge\Sigma\wedge$, by transforming it to a read-once oblivious ABP (ROABP); but it is inapplicable to our models with the top Π-gate (due to large waring rank and ROABP-width). A priori, our models are incomparable to ROABP, and thus the famous PIT algorithms for ROABP [32, 31, 41] are not expected to help either.

Similarly, a naive application of the *Jacobian* and *certifying path* technique from [5] fails for our models because it is difficult to come up with a *faithful* map for constant-variate reduction. Kumar and Saraf [59] crucially used that the computed polynomial has low individual degree (such that [26] can be invoked), while in [60] they exploits the low algebraic rank of the polynomials computed below the top Π-gate. Neither of them hold in general for our models. Very recently, Peleg and Shpilka [74] gave a poly-time blackbox PIT for $\Sigma^{[3]}\Pi\Sigma\Pi^{[2]}$, via incidence geometry (e.g. Edelstein-Kelly theorem involving 'quadratic' polynomials), by solving [39, Conj. 1] for $k = 3, \delta = 2$. The method seems very strenuous to generalize even to 'cubic' polynomials ($\delta = 3 = k$).

PIT for other models. Blackbox PIT algorithms for many restricted models are known. Egs. ROABP related models [75, 44, 3, 41, 42, 31, 8], log-variate circuits [30, 14], and non-commutative models [34, 61].

1.3. Techniques and motivation. Both the proofs are analytic as they use *logarithmic derivative*, and its power-series expansion which greatly transform the respective models. Where the nature of the first proof is inductive, the second is a more direct *one-shot* proof. In both the cases, we essentially reduce to the well-understood *wedge* models, that have unbounded top fanin, yet for which PITs are known. This reduction is unforeseeable and quite 'power'ful.

The analytic tool that we use, appears in algebra and complexity theory through the *formal power series* ring $\mathsf{R}[[x_1,\ldots,x_n]]$ (in short $\mathsf{R}[[\boldsymbol{x}]]$), see [70, 92, 22]. The advantages of the ring $\mathsf{R}[[\boldsymbol{x}]]$ are many and they usually emerge because of the inverse identity: $(1-x_1)^{-1} = \sum_{i\geq 0} x_1^i$, which does not make sense in $\mathsf{R}[x]$, but is valid in $\mathsf{R}[[\boldsymbol{x}]]$. Other analytic tools used are inspired from Wronskian (linear dependence) [55, Theorem 7] [50], Jacobian (algebraic dependence) [12, 5, 72], and logarithmic derivative operator $\mathsf{dlog}_{z_1}(f) = (\partial_{z_1} f)/f$.

We will be work with the division operator (e.g. $\mathsf{R}(z_1)$, over a certain ring R). However, the divisions do not come for free as they require invertibility with respect to z_1 throughout (again landing us in $\mathsf{R}[[z_1]]$. For circuit classes C, D we define class

$$\mathcal{C}/\mathcal{D} := \{f/g \mid f \in \mathcal{C}, \mathcal{D} \ni g \neq 0\}.$$

Similarly $\mathcal{C}\cdot\mathcal{D}$ to denotes the class taking respective products.

1.3.1. The DiDI-technique. In Theorem 1.1 we introduce a novel technique for designing PIT algorithms which comprises of inductively applying two fundamental operations on the input circuits to reduce it to a more tractable model. Suppose we want to test $\sum_{i\in[k]} T_i \stackrel{?}{=} 0$ where each T_i is computable by $\Pi\Sigma\wedge$. The idea is to *DI*vide it by T_k to obtain $1 + \sum_{i\in[k-1]} T_i/T_k$ and then *D*erivative to reduce the fanin to $k-1$ and obtain $\sum_{i\in[k-1]} \mathcal{T}_i$. Naturally, these operations pushes us to work with the fractional ring (e.g. $\mathsf{R}(z_1)$, over a certain ring R), further it also distorts the model as $\mathcal{T}_i$'s are no longer computable by simple $\Pi\Sigma\wedge$ circuits. However, with careful analytically analysis we establish that the non-zeroness is preserved in the reduced model. The process is then repeated until we reach $k = 1$, while maintaining the invariants which help us in preserving the non-zeroness till the end. We finish the proof by showing that the identity testing of reduced model can be done using known PIT algorithms.

1.3.2. Jacobian hits again. In Theorem 1.3 we exploit the prowess of the Jacobian polynomial first introduced in [12] and later explored in [5] to unify known PIT algorithms and design new ones. Suppose we want to test $\sum_{i\in[k]} T_i \stackrel{?}{=} 0$, where $T_i \in \Pi\Sigma\Pi^{[\delta]}$ (respec. $\Pi\Sigma\wedge$). We associate the Jacobian $J(T_1,\ldots,T_r)$ to captures the algebraic independence of $T_1,\ldots,T_r$ assuming this to be a transcendence basis of the T_i's. We design a variable reducing linear map Φ which preserves the algebraic independece of $T_1,\ldots,T_r$ and show that for any C: $C(T_1,\ldots,T_k) = 0 \iff C(\Phi(T_1),\ldots,\Phi(T_k)) = 0$. Such a map is called 'faithful' [5]. The map Φ ultimately provides a hitting set for $T_1 + \ldots + T_k$, as we reduce to a PIT of a polynomial over 'few' (roughly equal to k) variables, yielding a QP-time algorithm.

2. Preliminaries.

Before proving the results, we describe some of the assumptions and notations used throughout the paper. $\boldsymbol{x}$ denotes $(x_1,\ldots,x_n)$. $[n]$ denotes $\{1,\ldots,n\}$.

2.1. Notations and Definitions.

- **Logarithmic derivative.** Over a ring R and a variable y, the logarithmic derivative $\mathsf{dlog}_y : \mathsf{R}[y] \to \mathsf{R}(y)$ is defined as $\mathsf{dlog}_y(f) := \partial_y f/f$; here ∂_y denotes the partial derivative with respect to variable y. One important property of dlog is that it is additive over a product as

$$\mathsf{dlog}_y(f\cdot g) = \frac{\partial_y(f\cdot g)}{f\cdot g} = \frac{(f\cdot\partial_y g + g\cdot\partial_y f)}{f\cdot g} = \mathsf{dlog}_y(f) + \mathsf{dlog}_y(g).$$

  We refer this effect as *linearization* of product.
- **Circuit size.** Sparsity $\mathsf{sp}(\cdot)$ refers to the number of nonzero monomials. In this paper, it is a parameter of the circuit size. In particular, $\mathsf{size}(g_1\cdots g_s) = \sum_{i\in[s]}(\mathsf{sp}(g_i) + \deg(g_i))$, for $g_i \in \Sigma\wedge$ (respectively $\Sigma\Pi^{[\delta]}$). In whitebox settings, we also include the *bit-complexity* of the circuit (i.e. bit complexity of the constants used in the wires) in the size parameter. Some of the complexity parameters of a circuit are *depth* (number of layers), *syntactic degree* (the maximum degree polynomial computed by any node), *fanin* (maximum number of inputs to a node).
- **Hitting set.** A set of points $\mathcal{H} \subseteq \mathbb{F}^n$ is called a *hitting-set* for a class $\mathcal{C}$ of n-variate polynomials if for any nonzero polynomial $f \in \mathcal{C}$, there exists a point in $\mathcal{H}$ where f evaluates to a nonzero value. A $T(n)$-time hitting-set would mean that the hitting-set can be generated in time $T(n)$, for input size n.
- **Valuation.** Valuation is a map $\mathsf{val}_y : \mathsf{R}[y] \to \mathbb{Z}_{\geq 0}$, over a ring R, such that $\mathsf{val}_y(\cdot)$ is defined to be the maximum power of y dividing the element. It can be

easily extended to fraction field $\mathsf{R}(y)$, by defining $\mathsf{val}_y(p/q) := \mathsf{val}_y(p) - \mathsf{val}_y(q)$; where it can be negative.

- **Field.** We denote the underlying field as $\mathbb{F}$ and assume that it is of characteristic 0. All our results hold for other fields (eg. $\mathbb{Q}_p, \mathbb{F}_p$) of *large* characteristic (see Remarks in Section 3-4).
- **Jacobian.** The Jacobian of a set of polynomials $\mathbf{f} = \{f_1, \ldots, f_m\}$ in $\mathbb{F}[\boldsymbol{x}]$ is defined to be the matrix $\mathcal{J}_{\boldsymbol{x}}(\mathbf{f}) := \left(\partial_{x_j}(f_i)\right)_{m \times n}$. Let $S \subseteq \boldsymbol{x} = \{x_1, \ldots, x_n\}$ and $|S| = m$. Then, polynomial $J_S(\mathbf{f})$ denotes the minor (i.e. determinant of the submatrix) of $\mathcal{J}_{\boldsymbol{x}}(\mathbf{f})$, formed by the columns corresponding to the variables in S.

2.2. Basics of Algebraic Complexity Theory. For detailed discussion on the basics of Algebraic Complexity Theory we will encourage readers to refer [91, 82, 65, 83, 78]. Here we will formally state a few of the PIT results and properties of circuits for the later reference.

Trivial PIT Algorithm. The simplest PIT algorithm for any circuit in general is due to Polynomial Identity Lemma [71, 18, 95, 87]. When the number of variables is small, say $O(1)$, then this algorithm is very efficient.

LEMMA 2.1 (Trivial PIT). *For a class of n-variate, individual degree $< d$ polynomial $f \in \mathbb{F}[\boldsymbol{x}]$ there exists a deterministic PIT algorithm which runs in time $O(d^n)$.*

Sparse Polynomial. Sparse PIT is testing the identity of polynomials with bounded number of monomials. There have been a lot of work on sparse-PIT, interested readers can refer [13, 53] and references therein. For the proof of poly-time hitting set of Sparse PIT see [82, Thm. 2.1].

THEOREM 2.2 (Sparse-PIT map [53]). *Let $p(\boldsymbol{x}) \in \mathbb{F}[\boldsymbol{x}]$ with individual degree at most d and sparsity at most m. Then, there exists $1 \le r \le (mn \log d)^2$, such that*

$$p(y, y^d, \ldots, y^{d^{n-1}}) \neq 0, \bmod y^r - 1.$$

If p is computable by a size-s $\Sigma\Pi$ circuit, then there is a deterministic algorithm to test its identity which runs in time $\mathsf{poly}(s, m)$.

Indeed if identity of sparse polynomial can be tested efficiently, product of sparse polynomial can be tested efficiently. We formalise this in the following:

LEMMA 2.3 ([77] Lemma 2.3). *For a class of n-variate, degree d polynomial $f \in \mathbb{F}[\boldsymbol{x}]$ computable by $\Pi\Sigma\Pi$ of size s, there is a deterministic PIT algorithm which runs in time $\mathsf{poly}(s, d)$.*

A set $\mathcal{H} \subseteq \mathbb{F}^n$ is called a Hitting Set for a class polynomial $\mathcal{C} \subseteq \mathbb{F}[\boldsymbol{x}]$, if for all $g \in \mathcal{C}$

$$g \neq 0 \iff \exists \boldsymbol{\alpha} \in \mathcal{H} : g(\boldsymbol{\alpha}) \neq 0.$$

In literature, PIT has a close association with Hitting set as the two notions are provably equivalent (refer Lemma 3.2.9 and 3.2.10 [28]). Note that the set $\mathcal{H}$ works for every polynomial of the class. Instead of a PIT algorithm occasionally we will use such a set.

LEMMA 2.4 (Hitting Set of $\Pi\Sigma\wedge$). *For a class of n-variate, degree d polynomial $f \in \mathbb{F}[\boldsymbol{x}]$ computable by $\Pi\Sigma\Pi$ of size s, there is an explicit Hitting Set of size $\mathsf{poly}(s, d)$.*

Algebraic Branching Program (ABP). An ABP is a layered directed acyclic graph with $q+1$ many layers of vertices $V_0, \ldots, V_q$ with a source a and a sink b such that all the edges in the graph only go from a to V_0, V_{i-1} to V_i for any $i \in [q]$, and V_q to b. The edges have *uni*variate polynomials as their weights. The ABP is said to compute the polynomial

$$f(\boldsymbol{x}) \;=\; \sum_{p \in \mathsf{paths}(a,b)} \prod_{e \in p} W(e) \;,$$

where $W(e)$ is the weight of the edge e. The ABP has width-w if $|V_i| \leq w$, $\forall i \in \{0, \ldots, q\}$. In an equivalent definition, polynomials computed by ABP are of the form $A^T(\prod_{i\in[q]} D_i)B$, where $A, B \in \mathbb{F}^{w\times 1}[\boldsymbol{x}]$, and $D_i \in \mathbb{F}^{w\times w}[\boldsymbol{x}]$, where entries are univariate polynomials. We encourage interested readers to refer [91, 65] for more detailed discussion.

Definition 2.5 (Read-once oblivious ABP (ROABP)). *An ABP is called a* read-once oblivious ABP (ROABP) *if the edge weights are univariate polynomials in* distinct *variables across layers. Formally, there is a permutation π on the set $[q]$ such that the entries in the i-th matrix D_i are univariate polynomials over the variable $x_{\pi(i)}$, i.e., they come from the polynomial ring $\mathbb{F}[x_{\pi(i)}]$.*

A polynomial $f(x)$ is said to be computed by width-w ROABPs in *any order*, if for every permutation σ of the variables, there exists a width-w ROABP in the variable order σ that computes the polynomial $f(\boldsymbol{x})$. In whitebox setting, identity testing of any-order ROABP completely solved.

Theorem 2.6 (Theorem 2.4 [75]). *For n-variate polynomials computed by size-s ROABP, a hitting set of size $O(s^5 + s \cdot n^4)$ can be constructed.*

There have been quite a few results on blackbox PIT for ROABPs as well [32, 31, 41]. The current best known algorithm works in quasipolynomial time.

Theorem 2.7 (Theorem 4.9 [41]). *For n-variate, individual-degree-d polynomials computed by width-w ROABPs in any order, a hitting set of size $(ndw)^{O(\log\log w)}$ can be constructed.*

Depth-4 Circuits. A polynomial $f(\boldsymbol{x}) \in \mathbb{F}[\boldsymbol{x}]$ is computable by $\Sigma\wedge\Sigma\Pi^{[\delta]}$ circuits if $f(\boldsymbol{x}) = \sum_{i\in[s]} f_i(\boldsymbol{x})^{e_i}$ where $\deg f_i \leq \delta$. The first nontrivial PIT algorithm for this model was designed in [29].

Theorem 2.8 (Proposition 4.18 [29]). *There is a $\mathsf{poly}(n, d, \delta\log s)$-explicit hitting set of size $(nd)^{O(\delta\log s)}$ for the class of n-variate, degree-$(\leq d)$ polynomials $f(\boldsymbol{x})$, computed by $\Sigma\wedge\Sigma\Pi^{[\delta]}$-circuit of size s.*

Similarly, $\Sigma\wedge\Sigma\wedge$ circuits compute polynomials of the form $f(\boldsymbol{x}) = \sum_{i\in[s]} f_i^{e_i}$ where f_i is a sum of univariate polynomials. Using duality trick [81] and PIT results from [75, 41], one can design efficient PIT algorithm for $\Sigma\wedge\Sigma\wedge$ circuits.

Lemma 2.9 (PIT for $\Sigma\wedge\Sigma\wedge$-circuits). *Let $P \in \Sigma\wedge\Sigma\wedge$ of size s. Then, there exists a $\mathsf{poly}(s)$ (respectively $s^{O(\log\log s)}$) time whitebox (respectively blackbox) PIT for the same.*

Proof sketch. We show that any $g(\boldsymbol{x})^e = (g_1(x_1) + \ldots + g_n(x_n))^e$, where $\mathsf{deg}(g_i) \leq s$ can be written as $\sum_j h_{j1}(x_1)\cdots h_{jn}(x_n)$, for some $h_{j\ell} \in \mathbb{F}[x_\ell]$ of degree at most es. Define, $G := (y + g_1)\cdots(y + g_n) - y^n$. In its e-th power, notice that the leading-coefficient is $\mathsf{coef}_{y^{e(n-1)}}(G^e) = g^e$. So, interpolate on $e(n-1)+1$ many points ($y =$

$\beta_i \in \mathbb{F}$) to get

$$\mathsf{coef}_{y^{e(n-1)}}(G^e) \;=\; \sum_{i=1}^{e(n-1)+1} \alpha_i \, G^e(\beta_i)\;.$$

Now, expand $G^e(\beta_i) = ((\beta_i + g_1)\cdots(\beta_i + g_n) - \beta_i^n)^e$, by binomial expansion (without expanding the inner n-fold product). The top-fanin can be atmost $s \cdot (e+1) \cdot (e(n-1)+1) = O(se^2n)$. The individual degrees of the intermediate univariates can be at most es. Thus, it can be computed by an ROABP (of *any order*) of size at most $O(s^2e^3n)$.

Now, if $f = \sum_{j\in[s]} f_j^{e_j}$ is computed by a $\Sigma\wedge\Sigma\wedge$ circuit of size s, then clearly, f can also be computed by an ROABP (of any order) of size at most $O(s^6)$. So, the whitebox PIT follows from Theorem 2.6, while the blackbox PIT follows from Theorem Theorem 2.7. ∎

Further, $\Sigma\wedge\Sigma\wedge$ can be shown to be closed under multiplication i.e., product of two polynomials, each computable by a $\Sigma\wedge\Sigma\wedge$ circuit, is computable by a single $\Sigma\wedge\Sigma\wedge$ circuit. To prove that we will need an efficient way to write a product of a few powers as a sum of powers, using simple interpolation. For an algebraic proof, see [15, Proposition 4.3].

LEMMA 2.10 (Waring Identity for a monomial). *Let $M = x_1^{b_1}\cdots x_k^{b_k}$, where $1 \le b_1 \le \ldots \le b_k$, and* roots of unity *$\mathcal{Z}(i) := \{z \in \mathbb{C} : z^{b_i+1} = 1\}$. Then,*

$$M = \sum_{\varepsilon(i)\in\mathcal{Z}(i): i=2,\cdots,k} \gamma_{\varepsilon(2),\ldots,\varepsilon(k)} \cdot (x_1 + \varepsilon(2)x_2 + \ldots + \varepsilon(k)x_k)^d\;,$$

where $d := \deg(M) = b_1 + \ldots + b_k$, and $\gamma_{\varepsilon(2),\ldots,\varepsilon(k)}$ are scalars ($\mathsf{rk}(M) := \prod_{i=2}^k (b_i+1)$ many).

Remark. We actually need not work with $\mathbb{F} = \mathbb{C}$. We can go to a small extension (at most d^k), for a monomial of degree d, to make sure that $\varepsilon(i)$ exists.

Using the above lemma we prove the closure result.

LEMMA 2.11. *Let $f_i(\boldsymbol{x}, y) \in \mathbb{F}[y][\boldsymbol{x}]$, of syntactic degree $\le d_i$, be computed by a $\Sigma\wedge\Sigma\wedge$ circuit of size s_i, for $i \in [k]$ (wrt $\boldsymbol{x}$). Then, $f_1\cdots f_k$ has $\Sigma\wedge\Sigma\wedge$ circuit of size $O((d_2+1)\cdots(d_k+1)\cdot s_1\cdots s_k)$.*

Proof. Let $f_i = \sum_j f_{ij}^{e_{ij}}$; by assumption $e_{ij} \le d_i$ (by assumption). Then using Lemma 2.10, $f_{1j_1}^{e_{1j_1}}\cdots f_{kj_k}^{e_{kj_k}}$ has size at most $(d_2+1)\cdots(d_k+1)\cdot\left(\sum_{i\in[k]} \mathsf{size}(f_{ij_i})\right)$, for indices $j_1,\ldots,j_k$. Summing up for all $s_1\cdots s_k$ many products (atmost) gives the upper bound. ∎

3. Whitebox PIT for $\Sigma^{[k]}\Pi\Sigma\wedge$.

We consider a bloated model of computation which naturally generalizes $\Sigma\Pi\Sigma\wedge$ circuits and works ideally under the DiDI-techniques.

DEFINITION 3.1. *We call a circuit $\mathcal{C} \in \mathsf{Gen}(k,s)$, over $\mathsf{R}(\boldsymbol{x})$, for any ring R, with parameter k and size-s, if $\mathcal{C} \in \Sigma^{[k]}(\Pi\Sigma\wedge/\Pi\Sigma\wedge)\cdot(\Sigma\wedge\Sigma\wedge/\Sigma\wedge\Sigma\wedge)$. It computes $f \in \mathsf{R}(\boldsymbol{x})$, if $f = \sum_{i=1}^k T_i$, where*

- *$T_i =: (U_i/V_i)\cdot(P_i/Q_i)$, for $U_i, V_i \in \Pi\Sigma\wedge$, and $P_i, Q_i \in \Sigma\wedge\Sigma\wedge$,*
- *$\mathsf{size}(T_i) = \mathsf{size}(U_i) + \mathsf{size}(V_i) + \mathsf{size}(P_i) + \mathsf{size}(Q_i)$, and $\mathsf{size}(f) = \sum_{i\in[k]} \mathsf{size}(T_i)$.*

It is easy to see that all size-s $\Sigma^{[k]}\Pi\Sigma\wedge$ circuit are in $\mathsf{Gen}(k,s)$. We will design the *recursive* algorithm on $\mathsf{Gen}(k,s)$.

Proof of Theorem 1.1. Begin with defining $T_{i,0} := T_i$ and $f_0 := f$ where $T_{i,0} \in \Pi\Sigma\wedge$; $\sum_i T_{i,0} = f_0$, and f_0 has size $\le s$. Assume $\mathsf{deg}(f) < d \le s$; we keep the parameter d separately, to help optimize the complexity later. In every recursive call we work with $\mathsf{Gen}(\cdot,\cdot)$ circuits.

As the input case, define $U_{i,0} := T_{i,0}$ and $V_{i,0} := P_{i,0} := Q_{i,0} := 1$. We will use the hitting set of product of sparse polynomials (refer section 2.2) to obtain a point $\boldsymbol{\alpha} = (a_1, \dots, a_n) \in \mathbb{F}^n$ such that $U_{i,0}|_{\boldsymbol{x}=\boldsymbol{\alpha}} \neq 0$, for all $i \in [k]$. Eventually this evaluation point will help in maintaining the invertibility of $\Pi\Sigma\wedge$. Consider

$$g \;:=\; \prod_{i\in[k]} T_{i,0} \;=\; \prod_{i\in[k]} U_{i,0} \;=\; \prod_{i\in[\ell]} \sum_{j\in[n]} f_{ij}(x_j)\,,$$

where $f_{ij}(x_j)$ are univariate polynomials of degree at most d and $\ell \le k\cdot s$. Note that $\deg g \le d\cdot k\cdot s$ and g is computable by a $\Pi\Sigma\wedge$ circuit of size $O(s)$. Invoke Lemma 2.4 to obtain a hitting set $\mathcal{H}$, then evaluate g on every point of $\mathcal{H}$ to find an element $\boldsymbol{\alpha} \in \mathcal{H}$ such that $g(\boldsymbol{\alpha}) \neq 0$. We emphasise that in whitebox setting all $U_{i,0}$, are readily available for evaluation. Since, the size of the set is $\mathsf{poly}(s)$ and each evaluation takes $\mathsf{poly}(s)$ time, this preliminary step will add $\mathsf{poly}(s)$ time to the overall time complexity. Moreover, we obtain the $\boldsymbol{\alpha} \in \mathbb{F}^n$ which possess the required property.

To capture the non-zeroness, consider a 1-1 homomorphism $\Phi : \mathbb{F}[\boldsymbol{x}] \longrightarrow \mathbb{F}[\boldsymbol{x}, z_1]$ such that $x_i \mapsto z_1 \cdot x_i + a_i$ where a_i is the i-th coordinate of $\boldsymbol{\alpha}$, obtained earlier. Invertibility implies that $f_0 = 0 \iff \Phi(f_0) = 0$. Now we proceed with the recursive algorithm which first reduces the identity testing from top-fanin k to $k-1$. Note: $k = 1$ is trivial.

First Step: Efficient reduction from k to $k-1$. By assumption, $\sum_{i=1}^k T_{i,0} = f_0$ and $T_{k,0} \neq 0$. Apply Φ both sides, then divide and derive:

$$\begin{aligned}
\sum_{i\in[k]} T_{i,0} \;=\; f_0 \;\;&\iff\;\; \sum_{i\in[k]} \Phi(T_{i,0}) \;=\; \Phi(f_0)\\
&\iff\;\; \sum_{i\in[k-1]} \frac{\Phi(T_{i,0})}{\Phi(T_{k,0})} + 1 \;=\; \frac{\Phi(f_0)}{\Phi(T_{k,0})}\\
&\implies\;\; \sum_{i\in[k-1]} \partial_{z_1}\left(\frac{\Phi(T_{i,0})}{\Phi(T_{k,0})}\right) \;=\; \partial_{z_1}\left(\frac{\Phi(f_0)}{\Phi(T_{k,0})}\right)\\
&\iff\;\; \sum_{i=1}^{k-1} \frac{\Phi(T_{i,0})}{\Phi(T_{k,0})}\cdot \mathsf{dlog}\left(\frac{\Phi(T_{i,0})}{\Phi(T_{k,0})}\right) \;=\; \partial_{z_1}\left(\frac{\Phi(f_0)}{\Phi(T_{k,0})}\right). \qquad (3.1)
\end{aligned}$$

Define the following:

- $\mathsf{R}_1 := \mathbb{F}[z_1]/\langle z_1^d\rangle$. Note that, (3.1) holds over $\mathsf{R}_1(\boldsymbol{x})$.
- $\widetilde{T}_{i,1} := \Phi(T_{i,0})/\Phi(T_{k,0})\cdot \mathsf{dlog}(\Phi(T_{i,0})/\Phi(T_{k,0}))$, $\forall\, i \in [k-1]$.
- $f_1 := \partial_{z_1}(\Phi(f_0)/\Phi(T_{k,0}))$, over $\mathsf{R}_1(\boldsymbol{x})$.

Definability of $T_{i,1}$ and f_1. It is easy to see that these are well-defined terms. Here, we emphasize that we do not exactly compute/store $\widetilde{T}_{i,1}$ as a fraction where the degree in z_1 is $< d$; instead it is computed as an element in $\mathbb{F}(z_1, \boldsymbol{x})$, where z_1 is a formal variable. Formally, we compute $T_{i,1} \in \mathbb{F}(z_1, \boldsymbol{x})$, such that $\widetilde{T}_{i,1} = T_{i,1}$, over

$\mathsf{R}_1(\boldsymbol{x})$. We keep track of the degree of z_1 in $T_{i,1}$. Thus, $\sum_{i\in[k-1]} T_{i,1} = f_1$, over $\mathsf{R}_1(\boldsymbol{x})$.

The 'iff' condition. To show that our one step of DiDI has reduced the identity testing of $\mathsf{Gen}(k-1,\cdot)$, we need an $\iff$ condition. So far equality in (3.1) over $\mathsf{R}_1(\boldsymbol{x})$ is *one-sided*. Note that $f_1 \neq 0$ implies $\mathsf{val}_{z_1}(f_1) < d =: d_1$. By assumption, $\Phi(T_{k,0})$ is invertible over $\mathsf{R}_1(\boldsymbol{x})$. Further, $f_1 = 0$, over $\mathsf{R}_1(\boldsymbol{x})$, which implies –

1. Either, $\Phi(f_0)/\Phi(T_{k,0})$ is z_1-free. Then $\Phi(f_0)/\Phi(T_{k,0}) \in \mathbb{F}(\boldsymbol{x})$, which further implies it is in $\mathbb{F}$, because of the map Φ (z_1-free implies $\boldsymbol{x}$-free, by substituting $z_1 = 0$). Also, note that $f_0, T_{k,0} \neq 0$ implies $\Phi(f_0)/\Phi(T_{k,0})$ is a *nonzero* element in $\mathbb{F}$. Thus, it suffices to check whether $\Phi(f_0)|_{z_1=0} = \Psi(f_0)$ is non-zero or not.

2. Or, $\partial_{z_1}(\Phi(f_0)/\Phi(T_{k,0})) = z_1^{d_1} \cdot p$ where $p \in \mathbb{F}(z_1, \boldsymbol{x})$ s.t. $\mathsf{val}_{z_1}(p) \geq 0$. By simple power series expansion, one can show that $p \in \mathbb{F}(x)[[z_1]]$.

   LEMMA 3.2 (Valuation). *Consider $f \in \mathbb{F}(\boldsymbol{x}, y)$ such that $\mathsf{val}_y(f) \geq 0$. Then, $f \in \mathbb{F}(\boldsymbol{x})[[y]] \bigcap \mathbb{F}(\boldsymbol{x}, y)$.*

   Proof Sketch 3.3. Let $f = g/h$, where $g, h \in \mathbb{F}[\boldsymbol{x}, y]$. Now, $\mathsf{val}_y(f) \geq 0$, implies $\mathsf{val}_y(g) \geq \mathsf{val}_y(h)$. Let $\mathsf{val}_y(g) = d_1$ and $\mathsf{val}_y(h) = d_2$, where $d_1 \geq d_2 \geq 0$. Write $g = y^{d_1} \cdot \tilde{g}$ and $h = y^{d_2} \cdot \tilde{h}$. Write, $\tilde{h} = h_0 + h_1\, y + h_2\, y^2 + \ldots + h_d\, y^d$, for some d. Note that $h_0 \neq 0$. Thus,

$$\begin{aligned} f &= y^{d_1-d_2} \cdot \tilde{g}/(h_0 + h_1\, y + \ldots + h_d\, y^d) \\ &= y^{d_1-d_2} \cdot (\tilde{g}/h_0) \cdot (1 + (h_1/h_0)\, y + \ldots + (h_d/h_0)\, y^d)^{-1} \ \in \mathbb{F}(\boldsymbol{x})[[y]] \,. \end{aligned}$$

   The last conclusion follows by the inverse identity in the power-series ring.

   Hence, $\Phi(f_0)/\Phi(T_{k,0}) = z_1^{d_1+1} \cdot q$ where $q \in F(\boldsymbol{x})[[z_1]]$, i.e.

$$\Phi(f_0)/\Phi(T_{k,0}) \in \langle z_1^{d_1+1} \rangle_{\mathbb{F}(\boldsymbol{x})[[z_1]]} \implies \mathsf{val}_{z_1}(\Phi(f_0)) \geq d+1,$$

   a contradiction.

Conversely, it is obvious that $f_0 = 0$ implies $f_1 = 0$. Thus, we have proved the following

$$\sum_{i\in[k]} T_{i,0} \neq 0 \ \text{ over } \mathbb{F}[\boldsymbol{x}] \iff \sum_{i\in[k-1]} T_{i,1} \neq 0 \ \text{ over } \mathsf{R}_1(\boldsymbol{x}), \ \text{ or }, \ 0 \neq \Phi(f_0)|_{z_1=0} \in \mathbb{F} \,.$$

Eventually, we show that $T_{i,1} \in (\Pi\Sigma\wedge/\Pi\Sigma\wedge) \cdot (\Sigma\wedge\Sigma\wedge/\Sigma\wedge\Sigma\wedge)$, over $\mathsf{R}_1(\boldsymbol{x})$, with polynomial blowup in size (Claim 3.6). So, the above circuit is in $\mathsf{Gen}(k-1,\cdot)$, over $\mathsf{R}_1(\boldsymbol{x})$, which we recurse on to finally give the identity testing. The subsequent steps will be a bit more tricky:

Induction step. Assume that we are in the j-th step ($j \geq 1$). Our induction hypothesis assumes –

1. $\sum_{i\in[k-j]} T_{i,j} = f_j$, over $\mathsf{R}_j(\boldsymbol{x})$, where $\mathsf{R}_j := \mathbb{F}[z_1]/\langle z_1^{d_j} \rangle$, and $T_{i,j} \neq 0$.
2. $\mathsf{val}_{z_1}(T_{i,j}) \geq 0, \forall i \in [k-j]$.
3. Non-zero preserving iff condition

$$\begin{aligned} f \neq 0, \text{ over } \mathbb{F}[\boldsymbol{x}] \iff & f_j \neq 0, \ \text{over } \mathsf{R}_j(\boldsymbol{x}), \\ & \text{or} \bigvee_{i=0}^{j-1} ((f_i/T_{k-i,i})|_{z_1=0} \neq 0, \text{ over } \mathbb{F}(\boldsymbol{x})) \end{aligned}$$

4. Here, $T_{i,j} =: (U_{i,j}/V_{i,j}) \cdot (P_{i,j}/Q_{i,j})$, where $U_{i,j}, V_{i,j} \in \Pi\Sigma\wedge$, and $P_{i,j}, Q_{i,j} \in \Sigma\wedge\Sigma\wedge$, each in $\mathsf{R}_j[\boldsymbol{x}]$. Think of them being computed as $\mathbb{F}(z_1, \boldsymbol{x})$, with the degrees being tracked. Wlog, assume that $\mathsf{val}_{z_1}(T_{k-j,j})$ is the minimal among all $T_{i,j}$'s.
5. $U_{i,j}|_{z_1=0}, V_{i,j}|_{z_1=0} \in \mathbb{F}\backslash\{0\}$.

We follow as before without applying homomorphism any further. Note that the 'or condition' in the hypothesis 3 is similar to the $j = 0$ case except that there is no Φ: this is because $\Phi(f_0)|_{z_1=0} \neq 0 \iff \Phi(f_0/T_{k,0})|_{z_1=0} \neq 0$. This condition just separates the derivative from the constant-term.

Efficient reduction from $k-j$ to $k-j-1$. Let $\mathsf{val}_{z_1}(T_{i,j}) =: v_{i,j}$, for all $i \in [k-j]$. Note that

$$\min_i \mathsf{val}_{z_1}(T_{i,j}) = \min_i \mathsf{val}_{z_1}(P_{i,j}/Q_{i,j}) = v_{k-j,j}$$

since $\mathsf{val}_{z_1}(U_{i,j}) = \mathsf{val}_{z_1}(V_{i,j}) = 0$ (else we reorder). We remark that $0 \leq v_{i,j} < d_j$ for all i's in j-th step; upper-bound is strict, since otherwise $T_{i,j} = 0$ over $\mathsf{R}_j(x)$.

Similar to the first step, we divide with $T_{k-j,j}$ which has $\min \mathsf{val}$ and then derive:

$$\begin{aligned}
\sum_{i\in[k-j]} T_{i,j} \;=\; f_j \iff & \sum_{i\in[k-j-1]} T_{i,j}/T_{k-j,j} \;+\; 1 \;=\; f_j/T_{k-j,j} \\
\implies & \sum_{i\in[k-j-1]} \partial_{z_1}(T_{i,j}/T_{k-j,j}) \;=\; \partial_{z_1}(f_j/T_{k-j,j}) \\
\iff & \sum_{i=1}^{k-j-1} T_{i,j}/T_{k-j,j} \cdot \mathsf{dlog}(T_{i,j}/T_{k-j,j}) \;=\; \partial_{z_1}(f_j/T_{k-j,j}) \qquad (3.2)
\end{aligned}$$

Define the following:

- $\mathsf{R}_{j+1} := \mathbb{F}[z_1]/\langle z_1^{d_{j+1}} \rangle$, where $d_{j+1} := d_j - v_{k-j,j} - 1$.
- $\widetilde{T}_{i,j+1} := T_{i,j}/T_{k-j,j} \cdot \mathsf{dlog}(T_{i,j}/T_{k-j,j})$, $\forall\, i \in [k-j-1]$.
- $f_{j+1} := \partial_{z_1}(f_j/T_{k-j,j})$, over $\mathsf{R}_{j+1}(\boldsymbol{x})$.

We emphasize on the fact again that we do not exactly compute $\widetilde{T}_{i,j+1} \bmod z_1^{d_{j+1}}$; instead it is computed as a fraction in $\mathbb{F}(z_1, \boldsymbol{x})$, with formal z_1. Formally, we compute $T_{i,j+1} \in \mathbb{F}(z_1, \boldsymbol{x})$, such that $\widetilde{T}_{i,j+1} = T_{i,j+1}$, over $\mathsf{R}_{j+1}(\boldsymbol{x})$. We keep track of the degree of z_1 in $T_{i,j+1}$. Next, we will show that all the inductive hypotheses assumed hold in the j^{th} step as well.

Hypothesis (1): Definability of $T_{i,j+1}$ and f_{j+1}. By the minimal valuation assumption, it follows that $\mathsf{val}(f_j) \geq v_{k-j,j}$, and thus $\widetilde{T}_{i,j+1}$ and f_{j+1} are all well-defined over $\mathsf{R}_{j+1}(\boldsymbol{x})$. Note that, (3.2) holds over $\mathsf{R}_{j+1}(\boldsymbol{x})$ as $d_{j+1} < d_j$ (because, whatever identity holds true $\bmod z_1^{d_j}$ must hold $\bmod z_1^{d_{j+1}}$ as well). Hence, we must have $\sum_{i=1}^{k-j-1} \widetilde{T}_{i,j+1} = f_{j+1}$, over $\mathsf{R}_{j+1}(\boldsymbol{x})$ thus proving the induction hypothesis (1).

Hypothesis (2): Positivity of Valuation. Since we divide by the $\min \mathsf{val}$, by definition we immediately get $\mathsf{val}_{z_1}(T_{i,j+1}) \geq 0$ proving the hypothesis. Further, we claim that min val computation in DiDI is easy. For this, recall from the definition of valuation

$$\min_i \mathsf{val}_{z_1}(P_{i,j}/Q_{i,j}) = \min_i(\mathsf{val}_{z_1}(P_{i,j}) - \mathsf{val}_{z_1}(P_{i,j})).$$

Therefore, for $\min \mathsf{val}$ we compute $\mathsf{val}_{z_1}(P_{i,j})$ and $\mathsf{val}_{z_1}(Q_{i,j})$ for all $i \in [k-j]$.

Here is an important lemma which shows that coefficient of y^e of a polynomial $f(\boldsymbol{x}, y) \in \mathbb{F}[\boldsymbol{x}, y]$, computed by a $\Sigma\wedge\Sigma\wedge$ circuit, can be computed by a small $\Sigma\wedge\Sigma\wedge$ circuit.

Lemma 3.4 (Coefficient extraction). *Let $f(\boldsymbol{x}, y) \in \mathbb{F}[y][\boldsymbol{x}]$ be computed by a $\Sigma\wedge\Sigma\wedge$ circuit of size s and degree d. Then, $\mathsf{coef}_{y^e}(f) \in \mathbb{F}[\boldsymbol{x}]$ can be computed by a small $\Sigma\wedge\Sigma\wedge$ circuit of size $O(sd)$, over $\mathbb{F}[\boldsymbol{x}]$.*

Proof Sketch 3.5. Let, $f = \sum_i \alpha_i \cdot g_i^{e_i}$. Of course, $e_i \le s$ and $\mathsf{deg}_y(f) \le d$. Thus, write $f = \sum_{i=0}^{d} f_i \cdot y^i$, where $f_i \in \mathbb{F}[\boldsymbol{x}]$. We can interpolate on $d+1$-many distinct points $y \in \mathbb{F}$ and conclude that f_i has a $\Sigma\wedge\Sigma\wedge$ circuit of size at most $O(sd)$.

Using Lemma 3.4 we known $\mathsf{coef}_{z_1^e}(P_{i,j})$ and $\mathsf{coef}_{z_1^e}(Q_{i,j})$ are in $\Sigma\wedge\Sigma\wedge$ over $F[\boldsymbol{x}]$. We can keep track of z_1 degree and thus interpolate to find the minimum $e < d_j$ such that the computed coefficients are $\neq 0$, which gives the respective val.

Hypothesis (3): The 'iff' condition. The above (3.2) pioneers to reduce from $k-j$-summands to $k-j-1$. But we want a $\iff$ condition to efficiently reduce the identity testing. If $f_{j+1} \neq 0$, then $\mathsf{val}_{z_1}(f_{j+1}) < d_{j+1}$. Further, $f_{j+1} = 0$, over $\mathsf{R}_{j+1}(\boldsymbol{x})$ implies–

1. Either, $f_j/T_{k-j,j}$ is z_1-free. This implies it is in $\mathbb{F}(\boldsymbol{x})$. Now, if indeed $f_0 \neq 0$, then the computed $T_{i,j}$ as well as f_j must be non-zero over $\mathbb{F}(z_1, \boldsymbol{x})$, by induction hypothesis (as they are non-zero over $\mathsf{R}_j(\boldsymbol{x})$). However,

$$\left(\frac{T_{i,j}}{T_{k-j,j}}\right)\Bigg|_{z_1=0} = \left(\frac{U_{i,j} \cdot V_{k-j,j}}{U_{k-j,j} \cdot V_{i,j}}\right)\Bigg|_{z_1=0} \cdot \left(\frac{P_{i,j} \cdot Q_{k-j,j}}{P_{k-j,j} \cdot Q_{i,j}}\right)\Bigg|_{z_1=0}$$
$$\in \mathbb{F} \cdot \left(\frac{\Sigma\wedge\Sigma\wedge}{\Sigma\wedge\Sigma\wedge}\right).$$

Thus,
$$\frac{f_j}{T_{k-j,j}} \in \sum \mathbb{F} \cdot \left(\frac{\Sigma\wedge\Sigma\wedge}{\Sigma\wedge\Sigma\wedge}\right) \in \left(\frac{\Sigma\wedge\Sigma\wedge}{\Sigma\wedge\Sigma\wedge}\right).$$
Here we crucially use that $\Sigma\wedge\Sigma\wedge$ is closed under multiplication (Lemma 2.11). Thus, this identity testing can be done in poly-time (Lemma 2.9). For, detailed time-complexity and calculations, see Claim 3.6 and its subsequent paragraph.

2. Or, $\partial_{z_1}(f_j/T_{k-j,j}) = z_1^{d_{j+1}} \cdot p$, where $p \in \mathbb{F}(z_1, \boldsymbol{x})$ s.t. $\mathsf{val}_{z_1}(p) \ge 0$. By a simple power series expansion, one concludes that $p \in \mathbb{F}(\boldsymbol{x})[[z_1]]$ (Lemma 3.2). Hence, one concludes that
$$\frac{f_j}{T_{k-j,j}} \in \left\langle z_1^{d_{j+1}+1} \right\rangle_{\mathbb{F}(\boldsymbol{x})[[z_1]]} \implies \mathsf{val}_{z_1}(f_j) \ge d_j,$$
i.e. $f_j = 0$, over $\mathsf{R}_j(\boldsymbol{x})$.

Conversely, $f_j = 0$, over $\mathsf{R}_j(\boldsymbol{x})$, implies
$$\mathsf{val}_{z_1}(f_j) \ge d_j \implies \mathsf{val}_{z_1}\left(\partial_{z_1}\left(\frac{f_j}{T_{k-j,j}}\right)\right) \ge d_j - v_{k-j,j} - 1$$
$$\implies f_{j+1} = 0, \text{ over } \mathsf{R}_{j+1}(\boldsymbol{x}).$$

Thus, we have proved that $\sum_{i\in[k-j]} T_{i,j} \neq 0$ over $\mathsf{R}_j(\boldsymbol{x})$ iff
$$\sum_{i\in[k-j-1]} T_{i,j+1} \neq 0 \text{ over } \mathsf{R}_{j+1}(\boldsymbol{x}) \text{ , or , } 0 \neq \left(\frac{f_j}{T_{k-j,j}}\right)\Bigg|_{z_1=0} \in \mathbb{F}(\boldsymbol{x}) \text{ .}$$

Therefore induction hypothesis (3) holds.

Hypothesis (4): Size analysis. We will show that $T_{i,j+1} \in (\Pi\Sigma\wedge/\Pi\Sigma\wedge)\cdot(\Sigma\wedge\Sigma\wedge/\Sigma\wedge\Sigma\wedge)$, over $\mathsf{R}_{j+1}(\boldsymbol{x})$, with only polynomial blowup in size. Let $\mathsf{size}(T_{i,j}) \le s_j$, for $i \in [k-j]$, and $j \in [k]$. Note that, by assumption, $s_0 \le s$.

Claim 3.6 (Final size). *$T_{1,k-1} \in (\Pi\Sigma\wedge/\Pi\Sigma\wedge)\cdot(\Sigma\wedge\Sigma\wedge/\Sigma\wedge\Sigma\wedge)$ of size $s^{O(k7^k)}$, over $\mathsf{R}_{k-1}(\boldsymbol{x})$.*

Proof. Steps $j = 0$ and $j > 0$ are slightly different because of the Φ. However the main idea of using power-series is the same which eventually shows that $\mathsf{dlog}(\Sigma\wedge) \in \Sigma\wedge\Sigma\wedge$.

We first deal with $j = 0$. Let $A - z_1 \cdot B = \Phi(g) \in \Sigma\wedge$, for some $A \in \mathbb{F}$ and $B \in \mathsf{R}_1[\boldsymbol{x}]$. Note that $A \ne 0$ because of the map Ψ. Further, $\mathsf{size}(B) \le O(d\cdot\mathsf{size}(g))$, as a single monomial of the form x^e can produce $d+1$-many monomials. Over $\mathsf{R}_1(\boldsymbol{x})$,

$$\mathsf{dlog}(\Phi(g)) = -\frac{\partial_{z_1}(B\cdot z_1)}{A(1-\frac{B}{A}\cdot z_1)} = -\frac{\partial_{z_1}(B\cdot z_1)}{A}\cdot\sum_{i=0}^{d_1-1}\left(\frac{B}{A}\right)^i\cdot z_1^i\,. \tag{3.3}$$

B^i has a trivial $\wedge\Sigma\wedge$-circuit of size $O(d\cdot\mathsf{size}(g))$. Also, $\partial_{z_1}(B\cdot z_1)$ has a $\Sigma\wedge$-circuit of size at most $O(d\cdot\mathsf{size}(g))$. Using waring identity (Lemma 2.10), we get that each $\partial_{z_1}(B\cdot z_1)\cdot(B/A)^i\cdot z_1^i$ has size $O(i\cdot d\cdot\mathsf{size}(g))$, over $\mathsf{R}_1(\boldsymbol{x})$. Summing over $i \in [d_1-1]$, the overall size is at most $O(d_1^2\cdot d\cdot\mathsf{size}(g)) = O(d^3\cdot\mathsf{size}(g))$, as $d_0 = d_1 = d$.

For the j-th step, we emphasize that the degree could be larger than d. Assume that syntactic degree of denominator and numerator of $T_{i,j}$ (each in $\mathbb{F}[\boldsymbol{x},\boldsymbol{z}]$) are bounded by D_j (it is *not* d_j as seen above; this is to save on the trouble of mod-computation at each step). Of course, $D_0 < d \le s$.

For $j > 0$, the above summation in (3.3) is over $\mathsf{R}_j(\boldsymbol{x})$. However the degree could be D_j (possibly more than d_j) of the corresponding A and B. Thus, the overall size after the power-series expansion would be $O(D_j^2\cdot d\cdot\mathsf{size}(g))$.

Using Lemma 3.7, we can show that $\mathsf{dlog}(P_{i,j}) \in \Sigma\wedge\Sigma\wedge/\Sigma\wedge\Sigma\wedge$ (similarly for $Q_{i,j}$), of size $O(D_j^2\cdot s_j)$. Also $\mathsf{dlog}(U_{i,j}\cdot V_{k-j,j}) \in \sum\mathsf{dlog}(\Sigma\wedge)$, i.e. sum of action of dlog on $\Sigma\wedge$ (since dlog linearizes product); and it can be computed by the above formulation. Thus, $\mathsf{dlog}(T_{i,j}/T_{k-j,j})$ is a sum of 4-many $\Sigma\wedge\Sigma\wedge/\Sigma\wedge\Sigma\wedge$ of size at most $O(D_j^2\,s_j)$ and 1-many $\Sigma\wedge\Sigma\wedge$ of size $O(D_j^2 d_j s_j)$ (from the above power-series computation) [Note: we summed up the $\Sigma\wedge\Sigma\wedge$-expressions from $\mathsf{dlog}(\Sigma\wedge)$ together]. Additionally the syntactic degree of each denominator and numerator (of the $\Sigma\wedge\Sigma\wedge/\Sigma\wedge\Sigma\wedge$) is $O(D_j)$. We rewrite the 4 expressions (each of $\Sigma\wedge\Sigma\wedge/\Sigma\wedge\Sigma\wedge$) and express it as a single $\Sigma\wedge\Sigma\wedge/\Sigma\wedge\Sigma\wedge$ using waring identity (Lemma 2.11), with the size blowup of $O(D_j^{12}\,s_j^4)$; here the syntatic degree blowsup to $O(D_j)$. Finally we add the remaining $\Sigma\wedge\Sigma\wedge$ circuit (of size $O(D_j^3 s_j)$ and degree $O(dD_j)$) to get $O(s_j^5 D_j^{16} d)$. To bound this, we need to understand the degree bound D_j.

Finally we need to multiply $T_{i,j}/T_{k-j,j} \in (\Pi\Sigma\wedge/\Pi\Sigma\wedge)\cdot(\Sigma\wedge\Sigma\wedge/\Sigma\wedge\Sigma\wedge)$ where each $\Sigma\wedge\Sigma\wedge$ is a product of two $\Sigma\wedge\Sigma\wedge$ expression of size s_j and syntactic degree D_j; clubbed together owing a blowup of $O(D_j\cdot s_j^2)$. Hence multiplying it with $\Sigma\wedge\Sigma\wedge/\Sigma\wedge\Sigma\wedge$ expression obtained from dlog computation above gives size blowup of $s_{j+1} = s^7\cdot D_j^{O(1)}\cdot d$.

Computing $T_{i,j}/T_{k-j,j}$ increases the syntactic degree 'slowly'; which is much less than the size blowup. As mentioned before, the deg-blowup in dlog-computation is $O(dD_j)$ and in the clearing of four expressions, it is just $O(D_j)$. Thus, $D_{j+1} = O(dD_j) \implies D_j = d^{O(j)}$.

The recursion on the size is $s_{j+1} = s_j^7 \cdot d^{O(j)}$. Using $d \le s$ we deduce, $s_j = (sd)^{O(j \cdot 7^j)}$. In particular, s_{k-1}, size after $k-1$ steps is $s^{O(k \cdot 7^k)}$. This computation quantitatively establishes induction hypothesis (4). ∎

Hypothesis (5): Invertibility of $\Pi\Sigma\wedge$-circuits. For invertibility, we want to emphasise that the dlog compuation plays a crucial role here. In the following lemma we claim that the action $\mathsf{dlog}(\Sigma\wedge\Sigma\wedge) \in \Sigma\wedge\Sigma\wedge/\Sigma\wedge\Sigma\wedge$, is of poly-size.

LEMMA 3.7 (Differentiation). *Let $f(\boldsymbol{x}, y) \in \mathbb{F}[y][\boldsymbol{x}]$ be computed by a $\Sigma\wedge\Sigma\wedge$ circuit of size s and degree d. Then, $\partial_y(f)$ can be computed by a small $\Sigma\wedge\Sigma\wedge$ circuit of size $O(sd^2)$, over $\mathbb{F}[y][\boldsymbol{x}]$.*

Proof Sketch 3.8. Lemma 3.4 shows that each f_e has $O(sd)$ size circuit where $f = \sum_e f_e\, y^e$. Doing this for each $e \in [0, d]$ gives a blowup of $O(sd^2)$.

Similarly consider the action on $\Pi\Sigma\wedge$. We know dlog distributes the product additively, so it suffices to work with $\mathsf{dlog}(\Sigma\wedge)$; and earlier in Claim 3.6 we saw that $\mathsf{dlog}(\Sigma\wedge) \in \Sigma\wedge\Sigma\wedge$ of poly-size. Assuming these, we simplify

$$\frac{T_{i,j}}{T_{k-j,j}} = \frac{U_{i,j} \cdot V_{k-j,j}}{V_{i,j} \cdot U_{k-j,j}} \cdot \frac{P_{i,j} \cdot Q_{k-j,j}}{Q_{i,j} \cdot P_{k-j,j}},$$

and its dlog. Thus, using (3.2), $U_{i,(j+1)}$ grows to $U_{i,j} \cdot V_{k-j,j}$ (and similarly $V_{i,(j+1)}$). This also means: $U_{i,(j+1)}|_{z_1=0} \in \mathbb{F} \setminus \{0\}$ and thereby proving the hypothesis.

Final time complexity. The above proof actually shows that $T_{1,k-1}$ is in $\mathsf{Gen}(1, s^{O(k \cdot 7^k)})$ over $\mathsf{R}_{k-1}(\boldsymbol{x})$; and that the degree bound on z_1 (over $\mathbb{F}[z_1, \boldsymbol{x}]$, keeping denominator and numerator 'in place') is $D_{k-1} = d^{O(k)}$. We cannot directly use the identity testing algorithms of the constituent simpler models due to $\mathsf{R}_{k-1}(\boldsymbol{x})$. Moreover, using hypothesis (2) and Lemma 3.2 we know that $T_{1,k-1} \in \mathbb{F}(\boldsymbol{x})[[z_1]]$ and it suffices to do identity testing on the first term of the powerseries: $T_{1,k-1}|_{z_1=0}$ over $\mathbb{F}(\boldsymbol{x})$. Note that, hypothesis (5) guarantees that $\Pi\Sigma\wedge$ part remains non-zero on $z_1 = 0$ evaluation, however, $\Sigma\wedge\Sigma\wedge/\Sigma\wedge\Sigma\wedge$ may be undefined. For this, we keep track of z_1 degree of numerator and denominator, which will be polynomially bounded as seen in the discussion above. We can easily interpolate and cancel the z_1 power to make it work. Basically this shows that to test $T_{1,k-1}$ we need to test $z_1^e \cdot \Sigma\wedge\Sigma\wedge$ over $\mathbb{F}[\boldsymbol{x}]$ where $e \ge 0$ due to positive valuation. Whitebox PIT of $\Sigma\wedge\Sigma\wedge$ is in poly-time using Lemma 2.9, and testing z_1^e is possible using Lemma 2.1 with appropriate degree bound. The proof above is constructive: we calculate $U_{i,j+1}$ (and other terms) from $U_{i,j}$ explicitly. Gluing everything together we conclude this part can be done in $s^{O(k7^k)}$ time.

What remains is to test the $z_1 = 0$-part of induction hypothesis (3); it could *short-circuit* the recursion much before $j = k-1$. As we mentioned before, in this case, we need to do a PIT on $\Sigma\wedge\Sigma\wedge$ only. At the j-th step, when we substitute $z_1 = 0$, the size of each $T_{i,j}$ can be at most s_j (by definition). We need to do PIT on a simpler model: $\sum^{[k-j]} \mathbb{F} \cdot (\Sigma\wedge\Sigma\wedge/\Sigma\wedge\Sigma\wedge)$. We can clear out and express this as a single $\Sigma\wedge\Sigma\wedge/\Sigma\wedge\Sigma\wedge$ expression; with a size blowup of $s_j^{O(k-j)} \le (sd)^{O(j(k-j)7^j)}$. Since this case could short-circuit the recursion, to bound the final time complexity, we need to consider the j which maximizes the exponent.

LEMMA 3.9. *Let $k \in \mathbb{N}$, and $h(x) := x(k-x)7^x$. Then,* $\max_{i \in [k-1]} h(i) = h(k-1)$.

Proof Sketch 3.10. Differentiate to get $h'(x) = (k-x)7^x - x7^x + x(k-x)(\log 7)7^x = 7^x \cdot [x^2(-\log 7) + x(k\log 7 - 2) + k]$. It vanishes at

$$x = \left(\frac{k}{2} - \frac{1}{\log 7}\right) + \sqrt{\left(\frac{k}{2} - \frac{1}{\log 7}\right)^2 - \frac{k}{\log 7}}\,.$$

Thus, h is maximized at the integer $x = k-1$.

Therefore, $\max_{j\in[k-1]} j(k-j)7^j = (k-1)7^{k-1}$. Finally, use Lemma 2.9 for the base-case whitebox PIT. Thus, the final time complexity is $s^{O(k\cdot 7^k)}$.

Here we also remark that in $z_1 = 0$ substitution $\Sigma\wedge\Sigma\wedge/\Sigma\wedge\Sigma\wedge$ may be undefined. However, we keep track of z_1 degree of numerator and denominator, which will be polynomially bounded as seen in the discussion above. We can easily interpolate and cancel the z_1 power to make it work.

Bit complexity. It is routine to show that the bit-complexity is really what we claim. Initially, the given circuit has bit-complexity s. The main blowup happens due to the dlog-computation which is a poly-size blowup. We also remark that while using Lemma 2.11 (using Lemma 2.10), we *may* need to go to a field extension of at most $s^{O(k)}$ (because of the $\varepsilon(i)$ and correspondingly the constants $\gamma_{\varepsilon(2),\ldots,\varepsilon(k)}$, but they still are $s^{O(k)}$-bits). Also, Theorem 2.2 and Lemma 2.9 computations blowup bit-complexity polynomially. This concludes the proof. ∎

Remark 3.11.
1. The above method does *not* give whitebox PIT (in poly-time) for $\Sigma^{[k]}\Pi\Sigma\Pi^{[\delta]}$, as we donot know poly-time whitebox PIT for $\Sigma\wedge\Sigma\Pi^{[\delta]}$. However, the above methods do show that whitebox-PIT for $\Sigma^{[k]}\Pi\Sigma\Pi^{[\delta]}$ polynomially *reduces* to whitebox-PIT for $\Sigma\wedge\Sigma\Pi^{[\delta]}$.
2. DiDI-technique can be used to give whitebox PIT for the general bloated model $\mathsf{Gen}(k, s)$.
3. The above proof works when the characteristic is $\geq d$. This is because the nonzeroness remains *preserved* after derivation wrt z_1.

3.1. Algorithm. The whitebox PIT for Theorem 1.1, that is discussed in section 3, appears (below) as Algorithm 3.1.

Words of caution: Throughout the algorithm there are intermediate expressions to be stored compactly. Think of them as 'special' circuits in $\boldsymbol{x}$, but over the *function-field* $\mathbb{F}(\boldsymbol{z})$. Keep track of their degrees wrt z_1; and that of the sizes of their fractions represented in 'bloated' circuit form.

4. Blacbox PIT for Depth-4 Circuits. We will give the proof of Theorem 1.3 in this section. Before the details, we will state a few important definitions and lemmas from [5] to be referenced later.

DEFINITION 4.1 (Transcendence Degree). *Polynomials $T_1, \ldots, T_m$ are called* algebraically dependent *if there exists a nonzero* annihilator A *s.t.* $A(T_1, \ldots, T_m) = 0$. Transcendence degree *is the size of the largest subset $S \subseteq \{T_1, \ldots, T_m\}$ that is algebraically independent. Then S is called a* transcendence basis.

DEFINITION 4.2 (Faithful hom.). *A homomorphism $\Phi : \mathbb{F}[\boldsymbol{x}] \to \mathbb{F}[\boldsymbol{y}]$ is faithful for $\boldsymbol{T}$ if* $\mathsf{trdeg}_{\mathbb{F}}(\boldsymbol{T}) = \mathsf{trdeg}_{\mathbb{F}}(\Phi(\boldsymbol{T}))$.

The reason for interest in faithful maps is due its usefullness in preserve the identity as shown in the following fact.

FACT 4.3 (Theorem 2.4 [5]). *For any $C \in \mathbb{F}[y_1, \ldots, y_m]$, $C(\boldsymbol{T}) = 0 \iff C(\Phi(\boldsymbol{T})) = 0$.*

Algorithm 3.1 Whitebox PIT Algorithm for $\Sigma^{[k]}\Pi\Sigma\wedge$-circuits

INPUT: $f = T_1 + \ldots + T_k \in \Sigma^{[k]}\Pi\Sigma\wedge$**, a whitebox circuit of size** s **over** $\mathbb{F}[\boldsymbol{x}]$
OUTPUT: 0**, if** $f \equiv 0$**, and** 1**, if non-zero.**

Let $\Psi : \mathbb{F}[\boldsymbol{x}] \longrightarrow \mathbb{F}[z]$, be a sparse-PIT map, using [53] (Theorem 2.2). Apply it on f and check whether $\Psi(f) \stackrel{?}{=} 0$. If non-zero, `output` 1

Obtain a point $\boldsymbol{\alpha} = (a_1, \ldots, a_n) \in \mathbb{F}^n$ from Hitting Set $\mathcal{H}$ of $\Pi\Sigma\wedge$ such that $T_i|_{\boldsymbol{x}=\boldsymbol{\alpha}} \neq 0$, for all $i \in [k]$. And define $\Phi : x_i \mapsto z_1 \cdot x_i + a_i$. Check $\sum_{i\in[k-1]} \partial_{z_1}(\Phi(T_i)/\Phi(T_k)) \stackrel{?}{=} 0 \bmod z_1^{d_1}$ $(d_1 := s)$ as follows:

Consider each $T_{i,1} := \partial_{z_1}(\Phi(T_i)/\Phi(T_k))$ over $R_1(\boldsymbol{x})$, where $R_1 := \mathbb{F}[z_1]/\langle z_1^{d_1}\rangle$. Use dlog computation (Claim 3.6), to write each $T_{i,1}$ in a 'bloated' form as $(\Pi\Sigma\wedge/\Pi\Sigma\wedge)\cdot(\Sigma\wedge\Sigma\wedge/\Sigma\wedge\Sigma\wedge)$.

for $j \leftarrow 1$ **to** $k-1$ **do**

  Reduce the top-fanin at each step using '**Di**vide & **D**erive' technique. Assume that at j-th step, we have to check the identity: $\sum_{i\in[k-j]} T_{i,j} \stackrel{?}{=} 0$ over $R_j(\boldsymbol{x})$, where $R_j := \mathbb{F}[z_1]/\langle z_1^{d_j}\rangle$, each $T_{i,j}$ has a $(\Pi\Sigma\wedge/\Pi\Sigma\wedge)\cdot(\Sigma\wedge\Sigma\wedge/\Sigma\wedge\Sigma\wedge)$ representation and therein each $\Pi\Sigma\wedge|_{z_1=0} \in \mathbb{F}\setminus\{0\}$.

  Compute $v_{k-j,j} := \min_i \mathsf{val}_{z_1}(T_{i,j})$; by reordering it is for $i = k-j$. To compute $v_{k-j,j}$, use coefficient extraction (Lemma 3.4) and $\Sigma\wedge\Sigma\wedge$-circuit PIT (Lemma 2.9).

  '**D**ivide' by $T_{k-j,j}$ and check whether $\left.\left(\sum_{i\in[k-j-1]}(T_{i,j}/T_{k-j,j}) + 1\right)\right|_{z_1=0} \stackrel{?}{=} 0$.
  Note: this expression is in $(\Sigma\wedge\Sigma\wedge/\Sigma\wedge\Sigma\wedge)$. Use— (1) $\Pi\Sigma\wedge|_{z_1=0} \in \mathbb{F}$, and (2) *closure* of $\Sigma\wedge\Sigma\wedge$ under multiplication. Finally, do PIT on this by Lemma 2.9.

  If it is non-zero, `output` 1, otherwise '**D**erive' wrt z_1 and '**I**nduct' on $\left(\sum_{i\in[k-j-1]}\partial_{z_1}(T_{i,j}/T_{k-j,j})\right) \stackrel{?}{=} 0$, over $R_{j+1}(\boldsymbol{x})$ where $R_{j+1} := \mathbb{F}[z_1]/\langle z_1^{d_j - v_{k-j,j}-1}\rangle$.

  Again using dlog (Claim 3.6), show that $T_{i,j+1} := \partial_{z_1}(T_{i,j}/T_{k-j,j})$ has small $(\Pi\Sigma\wedge/\Pi\Sigma\wedge)\cdot(\Sigma\wedge\Sigma\wedge/\Sigma\wedge\Sigma\wedge)$-circuit over $R_{j+1}(\boldsymbol{x})$. So call the algorithm on $\sum_{i\in[k-j-1]} T_{i,j+1} \stackrel{?}{=} 0$.

  $j \leftarrow j+1$.

end for

At the end, $j = k-1$. Do PIT (Lemma 2.9) on the single $(\Pi\Sigma\wedge/\Pi\Sigma\wedge)\cdot(\Sigma\wedge\Sigma\wedge/\Sigma\wedge\Sigma\wedge)$ circuit, over $R_{k-1}(\boldsymbol{x})$. If it is zero, `output` 0 otherwise `output` 1.

Here is an important criterion about the jacobian matrix which basically shows that it *preserves* algabraic independence.

FACT 4.4 (Jacobian criterion). *Let* $\mathbf{f} \subset \mathbb{F}[\boldsymbol{x}]$ *be a finite set of polynomials of degree at most* d*, and* $\mathsf{trdeg}_{\mathbb{F}}(\mathbf{f}) \le r$*. If char*$(\mathbb{F}) = 0$*, or char*$(\mathbb{F}) > d^r$*, then* $\mathsf{trdeg}_{\mathbb{F}}(\mathbf{f}) = \mathsf{rk}_{\mathbb{F}(x)}\mathcal{J}_{\boldsymbol{x}}(\mathbf{f})$*.*

Jacobian criterion together with faithful maps give a recipe to design a map which drastically reduces number of variables, if trdeg is small.

LEMMA 4.5 (Lemma 2.7 [5]). *Let* $\boldsymbol{T} \in \mathbb{F}[\boldsymbol{x}]$ *be be a finite set of polynomials of degree at most* d *and* $\mathsf{trdeg}_{\mathbb{F}}(\boldsymbol{T}) \le r$*, and char(F)=0 or* $> d^r$*. Let* $\Psi' : \mathbb{F}[\boldsymbol{x}] \longrightarrow \mathbb{F}[z_1]$

such that $\mathsf{rk}_{\mathbb{F}(\boldsymbol{x})}\mathcal{J}_{\boldsymbol{x}}(\boldsymbol{T}) = \mathsf{rk}_{\mathbb{F}(z_1)}\Psi'(\mathcal{J}_{\boldsymbol{x}}(\boldsymbol{T}))$*.*

Then, the map $\Phi : \mathbb{F}[\boldsymbol{x}] \longrightarrow \mathbb{F}[z_1, t, \boldsymbol{y}]$*, such that* $x_i \mapsto (\sum_j y_j t^{ij}) + \Psi'(x_i)$*, is a faithful homomorphism for* $\boldsymbol{T}$*.*

In the next section we will use these tools to prove Theorem 1.3(b). The proof and calculations for Theorem 1.3(a) are very similar.

4.1. PIT for $\Sigma^{[k]}\Pi\Sigma\Pi^{[\delta]}$. We solve the PIT for a more general model than $\Sigma^{[k]}\Pi\Sigma\Pi$ by solving the following problem.

Problem 4.6. *Let* $\{T_i \,|\, i \in [m]\}$ *be* $\Pi\Sigma\Pi^{[\delta]}$ *circuits of (syntactic) degree at most* d *and size* s*. Let the transcendence degree of* T_i*'s,* $\mathsf{trdeg}_{\mathbb{F}}(T_1, \ldots, T_m) = k \ll s$*. Further,* $C(x_1, \ldots, x_m)$ *be a circuit of* $(\mathsf{size} + \deg) < s'$*. Design a blackbox-PIT algorithm for* $C(T_1, \ldots, T_m)$*.*

Trivially, $\Sigma^{[k]}\Pi\Sigma\Pi^{[\delta]}$ is a very special case of the above setting. Let $\boldsymbol{T} := \{T_1, \ldots, T_m\}$. Let $\boldsymbol{T}_k := \{T_1, \ldots, T_k\}$ be a transcendence basis. For $T_i = \prod_j g_{ij}$, we denote the set $L(T_i) := \{g_{ij} \mid j\}$.

We want to find an explicit homomorphism $\Psi : \mathbb{F}[\boldsymbol{x}] \to \mathbb{F}[\boldsymbol{x}, z_1]$ s.t. $\Psi(\mathcal{J}_{\boldsymbol{x}}(\boldsymbol{T}))$ is of a 'nice' form. In the image we fix $\boldsymbol{x}$ suitably, to get a composed map $\Psi' : \mathbb{F}[\boldsymbol{x}] \longrightarrow \mathbb{F}[z_1]$ s.t. $\mathsf{rk}_{\mathbb{F}(\boldsymbol{x})}\mathcal{J}_{\boldsymbol{x}}(\boldsymbol{T}) = \mathsf{rk}_{\mathbb{F}(z_1)}\Psi'(\mathcal{J}_{\boldsymbol{x}}(\boldsymbol{T}))$. Then, we can extend this map to $\Phi : \mathbb{F}[\boldsymbol{x}] \longrightarrow \mathbb{F}[z_1, \boldsymbol{y}, t]$ s.t. $x_i \mapsto (\sum_{j=1}^{k} y_j t^{ij}) + \Psi'(x_i)$, which is *faithful* Theorem 4.5. We show that the map Φ can be efficiently constructed using a scaling and shifting map (Ψ) which is eventually fixed by the hitting set (H' defining Ψ') of a $\Sigma\wedge\Sigma\Pi^{[\delta]}$ circuit. Overall, $\Phi(f)$ is a $k+2$-variate polynomial for which a trivial hitting set exists.

Wlog, $\mathcal{J}_{\boldsymbol{x}}(\boldsymbol{T})$ is full rank with respect to the variable set $\boldsymbol{x}_k = (x_1, \ldots, x_k)$. Thus, by assumption, $J_{\boldsymbol{x}_k}(\boldsymbol{T}_k) \neq 0$ (for notation, see section 2). We want to construct a Ψ s.t. $\Psi(J_{\boldsymbol{x}_k}(\boldsymbol{T}_k))$ has an 'easier' PIT. We have the following identity [5, Eqn. 3.1], from the linearity of the determinant, and the simple observation that $\partial_x(T_i) = T_i \cdot \left(\sum_j \partial_x(g_{ij})/g_{ij}\right)$, where $T_i = \prod_j g_{ij}$:

$$J_{\boldsymbol{x}_k}(\boldsymbol{T}_k) = \sum_{g_1 \in L(T_1), \ldots, g_k \in L(T_k)} \left(\frac{T_1 \ldots T_k}{g_1 \ldots g_k}\right) \cdot J_{\boldsymbol{x}_k}(g_1, \ldots, g_k)\,. \tag{4.1}$$

The homomorphism Ψ. To ensure the invertibility of all $g \in \bigcup_i L(T_i)$ we proceed as in section 3. Consider

$$h := \prod_{i\in[k]} \prod_{g \in L(T_i)} g = \prod_{i \in [\ell]} g,$$

where $g \in \bigcup_i L(T_i)$ and $\ell \leq k \cdot s$. Note that $\deg h \leq d \cdot k \cdot s$ and h is computable by $\Pi\Sigma\Pi$ circuit of size $O(s)$. Theorem 2.4 gives the relevant hitting set $\mathcal{H} \subseteq \mathbb{F}^n$ which contains an evaluation point $\boldsymbol{\alpha} = (a_1, \ldots, a_n)$ such that $h(\boldsymbol{\alpha}) \neq 0$ implying $g(\boldsymbol{\alpha}) \neq 0$, for all $g \in \bigcup_i L(T_i)$. We emphasise that, unlike the previous case, here in the blackbox setting, we *do not* have individual access of g to verify for the correct $\boldsymbol{\alpha}$. Thus, we try out all $\boldsymbol{\alpha} \in \mathcal{H}$ to see whichever works. If the input polynomial f is non-zero, then one such $\boldsymbol{\alpha}$ must exist. This search adds a multiplicative blowup of $\mathsf{poly}(s)$, since the size of $\mathcal{H}$ is $\mathsf{poly}(s)$.

Fix an $\boldsymbol{\alpha} = (a_1, \cdots, a_n) \in \mathcal{H}$ and define $\Psi : \mathbb{F}[\boldsymbol{x}] \to \mathbb{F}[\boldsymbol{x}, z_1]$ as $x_i \mapsto z_1 \cdot x_i + a_i$. Denote the ring $\mathsf{R}[\boldsymbol{x}]$ where $\mathsf{R} := \mathbb{F}[z_1]/\langle z_1^D \rangle$, and $D := k \cdot (d-1) + 1$. Being 1-1, Ψ is clearly a non-zero preserving map. Moreover, CLAIM 4.7. *$J_{\boldsymbol{x}_k}(\boldsymbol{T}_k) = 0 \iff \Psi(J_{\boldsymbol{x}_k}(\boldsymbol{T}_k)) = 0$, over $\mathsf{R}[\boldsymbol{x}]$.*

Proof. As $\mathsf{deg}(T_i) \le d$, each entry of the matrix can be of degree at most $d-1$; therefore $\mathsf{deg}(J_{\boldsymbol{x}_k}(\boldsymbol{T}_k)) \le k(d-1) = D-1$. Thus, $\mathsf{deg}_{z_1}(\Psi(J_{\boldsymbol{x}_k}(\boldsymbol{T}_k))) < D$. Hence, the conclusion. ∎

Equation 4.1 implies that

$$\Psi(J_{\boldsymbol{x}_k}(\boldsymbol{T}_k)) \;=\; \Psi(T_1 \cdots T_k) \cdot \sum_{g_1 \in L(T_1), \ldots, g_k \in L(T_k)} \frac{\Psi(J_{\boldsymbol{x}_k}(g_1, \ldots, g_k))}{\Psi(g_1 \ldots g_k)} \,. \tag{4.2}$$

As T_i has product fanin s, the top-fanin in the sum in Equation 4.2 can be at most s^k. Then define,

$$\widetilde{F} \;:=\; \sum_{g_1 \in L(T_1), \ldots, g_k \in L(T_k)} \frac{\Psi(J_{\boldsymbol{x}_k}(g_1, \ldots, g_k))}{\Psi(g_1 \ldots g_k)} \;, \quad \text{over } \mathsf{R}[\boldsymbol{x}]. \tag{4.3}$$

Well-definability of $\widetilde{F}$. Note that,

$$\Psi(g_i) \equiv \Psi_1(g_i) \bmod z_1 \ne 0 \implies 1/\Psi(g_1 \cdots g_k) \in \mathbb{F}[[\boldsymbol{x}, z_1]].$$

Thus, RHS is an element in $\mathbb{F}[[\boldsymbol{x}, z_1]]$ and taking mod z_1^D it is in $\mathsf{R}[\boldsymbol{x}]$. We remark that instead of minimally reducing mod z_1^D, we will work with an $F \in \mathbb{F}[z_1, \boldsymbol{x}]$ such that $F = \tilde{F}$ over $\mathsf{R}[\boldsymbol{x}]$. Further, we ensure that the degree of z_1 is polynomially bounded.

CLAIM 4.8. *Over $\mathsf{R}[\boldsymbol{x}]$, $\Psi(J_{\boldsymbol{x}_k}(\boldsymbol{T}_k)) = 0 \iff F = 0$.*

Proof sketch. This follows from the invertibility of $\Psi(T_1 \cdots T_k)$ in $R[\boldsymbol{x}]$. ∎

The hitting set H'. By $J_{\boldsymbol{x}_k}(\boldsymbol{T}_k) \ne 0$, and Claims 4.7-4.8, we have $F \ne 0$ over $\mathsf{R}[\boldsymbol{x}]$. We want to find $H' \subseteq \mathbb{F}^n$, s.t. $\Psi(J_{\boldsymbol{x}_k}(\boldsymbol{T}_k))|_{\boldsymbol{x}=\boldsymbol{\alpha}} \ne 0$, for some $\boldsymbol{\alpha} \in H'$ (which will ensure the rank-preservation). Towards this, we will show (below) that F has $s^{O(\delta k)}$-size $\Sigma\wedge\Sigma\Pi^{[\delta]}$-circuit over $\mathsf{R}[\boldsymbol{x}]$. Next, Theorem 2.8 provides the hitting set H' in time $s^{O(\delta^2 k \log s)}$.

CLAIM 4.9 (Main size bound). *$F \in \mathsf{R}[\boldsymbol{x}]$ has $\Sigma\wedge\Sigma\Pi^{[\delta]}$-circuit of size $(s3^\delta)^{O(k)}$.*

The proof studies the two parts of Equation 4.3—

1. The numerator $\Psi(J_{\boldsymbol{x}_k}(g_1, \ldots, g_k))$ has $O(3^\delta 2^k k! ks)$-size $\Sigma\wedge\Sigma\Pi^{[\delta-1]}$-circuit (see Theorem 4.14), and
2. $1/\Psi(g_1 \cdots g_k)$, for $g_i \in L(T_i)$ has $(s3^\delta)^{O(k)}$-size $\Sigma\wedge\Sigma\Pi^{[\delta]}$-circuit; both over $\mathsf{R}[\boldsymbol{x}]$ (see Theorem 4.15).

We need the following two claims to prove the numerator size bound.

CLAIM 4.10. *Let $g_i \in L(T_i)$, where $T_i \in \Pi\Sigma\Pi^{[\delta]}$ of size atmost s, then the polynomial $J_{\boldsymbol{x}_k}(g_1, \ldots, g_k)$ is computable by $\Sigma^{[k!]}\Pi^{[k]}\Sigma\Pi^{[\delta-1]}$ of size $O(k!\,ks)$.*

Proof Sketch 4.11. Each entry of the matrix has degree at most $\delta - 1$. Trivial expansion gives $k!$ top-fanin where each product (of fanin k) has size $\sum_i \mathsf{size}(g_i)$. As, $\mathsf{size}(T_i) \le s$, trivially each $\mathsf{size}(g_i) \le s$. Therefore, the total size is $k! \cdot \sum_i \mathsf{size}(g_i) = O(k!\,ks)$.

CLAIM 4.12. *Let $g \in \Sigma\Pi^\delta$, then $\Psi(g) \in \Sigma\Pi^\delta$ of size $3^\delta \cdot \mathsf{size}(g)$ (for $n \gg \delta$).*

Proof Sketch 4.13. Each monomial $\boldsymbol{x}^{\boldsymbol{e}}$ of degree δ, can produce $\prod_i(e_i + 1) \le ((\sum_i e_i + n)/n)^n \le (\delta/n + 1)^n$-many monomials, by AM-GM inequality as $\sum_i e_i \le \delta$. As $\delta/n \to 0$, we have $(1 + \delta/n)^n \to e^\delta$. As $e < 3$, the upper bound follows.

Lemma 4.14 (Numerator size). *$\Psi(J_{\boldsymbol{x}_k}(g_1,\ldots,g_k))$ is computable by $\Sigma\wedge\Sigma\Pi^{[\delta-1]}$ of size $O(3^\delta\, 2^k k\, k!s) =: s_2$.*

Proof. In Theorem 4.10 we showed that $J_{\boldsymbol{x}_k}(g_1,\ldots,g_k) \in \Sigma^{[k!]}\Pi^{[k]}\Sigma\Pi^{[\delta-1]}$ of size $O(k!ks)$. Moreover, for a $g \in \Sigma\Pi^{[\delta-1]}$, we have $\Psi(g) \in \Sigma\Pi^{[\delta-1]}$ of size at most $3^\delta \cdot \mathsf{size}(g)$, over $\mathsf{R}[\boldsymbol{x}]$ due to Theorem 4.12).

Combining these, one concludes that $\Psi(J_{\boldsymbol{x}_k}(g_1,\ldots,g_k)) \in \Sigma^{[k!]}\Pi^{[k]}\Sigma\Pi^{[\delta-1]}$, of size $O(3^\delta\, k!ks)$. We *convert* the Π-gate to $\wedge$ gate using waring identity (Theorem 2.10) which blowsup the size by a multiple of 2^{k-1}. Thus, $\Psi(J_{\boldsymbol{x}_k}(g_1,\ldots,g_k)) \in \Sigma\wedge\Sigma\Pi^{[\delta-1]}$ of size $O(3^\delta\, 2^k k\, k!s)$. ∎

In the following lemma, using power series expansion of expressions like $1/(1-a\cdot z_1)$, we conclude that $1/\Psi(g)$ has a small $\Sigma\wedge\Sigma\Pi^{[\delta]}$-circuit, which would further imply the same for $1/\Psi(g_1\cdots g_k)$.

Lemma 4.15 (Denominator size). *Let $g_i \in L(T_i)$. Then, $1/\Psi(g_1\cdots g_k)$ can be computed by a $\Sigma\wedge\Sigma\Pi^{[\delta]}$-circuit of size $s_1 := (s3^\delta)^{O(k)}$, over $\mathsf{R}[\boldsymbol{x}]$.*

Proof. Let $g \in L(T_i)$ for some i. Assume, $\Psi(g) = A - z_1\cdot B$, for some $A \in \mathbb{F}$ and $B \in \mathsf{R}[\boldsymbol{x}]$ of degree δ, with $\mathsf{size}(B) \le 3^\delta \cdot s$, from Theorem 4.12. Note that, over $\mathsf{R}[\boldsymbol{x}]$,

$$\frac{1}{\Psi(g)} \;=\; \frac{1}{A(1-\frac{B}{A}\cdot z_1)} \;=\; \frac{1}{A}\cdot\sum_{i=0}^{D-1}\left(\frac{B}{A}\right)^i \cdot z_1^i\,. \tag{4.4}$$

As, $\mathsf{size}(B^i)$ has a trivial $\wedge\Sigma\Pi^{[\delta]}$-circuit (over $\mathsf{R}[\boldsymbol{x}]$) of size $\le 3^\delta\cdot s + i$; summing over $i \in [D-1]$, the overall size is at most $D\cdot 3^\delta\cdot s + O(D^2)$. As $D < k\cdot d$, we conclude that $1/\Psi(g)$ has $\Sigma\wedge\Sigma\Pi^{[\delta]}$ of size $\mathsf{poly}(s\cdot k\cdot d3^\delta)$, over $\mathsf{R}[\boldsymbol{x}]$. Multiplying k-many such products directly gives an upper bound of $(s\cdot 3^\delta)^{O(k)}$, using Theorem 2.11 (basically, waring identity). ∎

Proof of Theorem 4.9. Combining Lemmas 4.14-4.15, observe that $\Psi(J_{\boldsymbol{x}_k}(\cdot)/\Psi(\cdot)$ has $\Sigma\wedge\Sigma\Pi^{[\delta]}$-circuit of size at most $(s_1\cdot s_2)^2 = (s\cdot 3^\delta)^{O(k)}$, over $\mathsf{R}[\boldsymbol{x}]$, using Theorem 2.11. Summing up at most s^k many terms (by defn. of F), the size still remains $(s\cdot 3^\delta)^{O(k)}$. ∎

Degree bound. As, syntactic degree of T_i are bounded by d, and Ψ maintain $\mathsf{deg}_{\boldsymbol{x}} = \mathsf{deg}_{z_1}$, we must have $\mathsf{deg}_{z_1}(\Psi(J_{\boldsymbol{x}_k}(g_1,\ldots,g_k)) = \mathsf{deg}_{\boldsymbol{x}}(J_{\boldsymbol{x}_k}(g_1,\ldots,g_k)) \le D-1$. Note that, Theorem 4.14 actually works over $\mathbb{F}[\boldsymbol{x},z_1]$ and thus there is no additional degree-blow up (in z_1). However, there is some degree blowup in Theorem 4.15, due to Equation 4.4.

Note that Equation 4.4 shows that over $\mathsf{R}[\boldsymbol{x}]$,

$$\frac{1}{\Psi(g)} = \left(\frac{1}{A^D}\right)\cdot\left(\sum_{i=0}^{D-1} A^{D-1-i} z_1^i \cdot B^i\right) =: \frac{p(\boldsymbol{x},z_1)}{q},$$

where $q = A^D$. We think of $p \in \mathbb{F}[\boldsymbol{x},z_1]$ and $q \in \mathbb{F}$. Note, $\mathsf{deg}_{z_1}(\Psi(g)) \le \delta$ implies $\mathsf{deg}_{z_1}(p) \le \mathsf{deg}_{z_1}((B\,z_1)^{D-1}) \le \delta\cdot(D-1)$.

Finally, denote $1/\Psi(g_1\cdots g_k) =: P_{g_1,\ldots,g_k}/Q_{g_1,\ldots,g_k}$, over $\mathsf{R}[\boldsymbol{x}]$. This is just multiplying k-many (p/q)'s; implying a degree blowup by a multiple of k. In particular – $\mathsf{deg}_{z_1}(P_{(\cdot)}) \le \delta\cdot k\cdot(D-1)$ Thus, in Equation 4.3, summing up s^k-many terms gives an expression (over $\mathsf{R}[\boldsymbol{x}]$):

$$F \;=\; \sum_{g_1\in L(T_1),\ldots,g_k\in L(T_k)} \Psi(J_{\boldsymbol{x}_k}(g_1,\ldots,g_k))\cdot\left(\frac{P_{g_1,\ldots,g_k}}{Q_{g_1,\ldots,g_k}}\right) \;=:\; \frac{P(\boldsymbol{x},z_1)}{Q}\,.$$

Verify that $Q \in \mathbb{F}$. The degree of z_1 also remains bounded by

$$\max_{g_i \in L(T_i), i \in [k]} \mathsf{deg}_{z_1}(P_{g_1,\ldots,g_k}) + \delta k \le \mathsf{poly}(s).$$

Using the degree bounds, we finally have $P \in \mathbb{F}[\boldsymbol{x}, z_1]$ as a $\Sigma \wedge \Sigma\Pi^{[\delta]}$-circuit (over $\mathbb{F}(z_1)$) of size $n^{O(\delta)}\,(s3^{\delta})^{O(k)} = 3^{O(\delta k)} s^{O(k+\delta)} =: s_3$.

We want to *construct* a set $H' \subseteq \mathbb{F}^n$ such that the action $P(H', z_1) \ne 0$. Using [29] (Theorem 2.8), we conclude that it has $s^{O(\delta \log s_3)} = s^{O(\delta^2 k \log s)}$ size hitting set which is constructible in a similar time. Hence, the construction of Φ follows, making $\Phi(f)$ a $k+2$ variate polynomial. Finally, by the obvious degree bounds of $\boldsymbol{y}, z_1, t$ from the definition of Φ, we get the blackbox PIT algorithm with time-complexity $s^{O(\delta^2 k \log s)}$; finishing Theorem 1.3(b).

We could also give the final hitting set for the general problem.

Solution to Theorem 4.6. We know that

$$C(T_1, \ldots, T_m) = 0 \iff E := \Phi(C(T_1, \ldots, T_m)) = 0.$$

Since, H' can be constructed in $s^{O(\delta^2\, k\, \log s)}$-time, it is trivial to find hitting set for $E|_{H'}$ (which is just a $k+2$-variate polynomial with the aformentioned degree bounds). The final hitting set for E can be constructed in $s'^{O(k)} \cdot s^{O(\delta^2\, k\, \log s)}$-time. ◻

Remark 4.16.

1. As Jacobian Criterion (Theorem 4.4) holds when the characteristic is $> d^{\mathsf{trdeg}}$, it is easy to conclude that our theorem holds for all fields of char $> d^k$.
2. The above proof gives an efficient reduction from blackbox PIT for $\Sigma^{[k]}\Pi\Sigma\Pi^{[\delta]}$ circuits to $\Sigma\wedge\Sigma\Pi^{[\delta]}$ circuits. In particular, a poly-time hitting set for $\Sigma\wedge\Sigma\Pi^{[\delta]}$ circuits would put PIT for $\Sigma^{[k]}\Pi\Sigma\Pi^{[\delta]}$ in P.
3. Also, DiDI-technique (of Theorem Theorem 1.1) directly gives a blackbox algorithm, but the complexity is *exponentially* worse (in terms of k in the exponent) for its recursive blowups.

4.2. PIT for $\Sigma^{[k]}\Pi\Sigma\wedge$. As we remarked earlier, the proof of Theorem 1.3(a) is similar to the one we discussed in section 4.1. Here we sketch the proof, stating some relevant changes. Similar to Theorem 1.3(b), we generalize this theorem and prove for a much bigger class of polynomials.

PROBLEM 4.17. *Let $\{T_i \,|\, i \in [m]\}$ be $\Pi\Sigma\wedge$ circuits of (syntactic) degree at most d and size s. Let the transcendence degree of T_i's, $\mathsf{trdeg}_{\mathbb{F}}(T_1, \ldots, T_m) =: k \ll s$. Further, $C(x_1, \ldots, x_m)$ be a circuit of size + degree $< s'$. Design a blackbox-PIT algorithm for $C(T_1, \ldots, T_m)$.*

It is trivial to see that $\Sigma^{[k]}\Pi\Sigma\wedge$ is a very special case of the above settings. We will use the same idea (& notation) as in Theorem 1.3(b), using the Jacobian technique. The main idea is to come up with Ψ map, and correspondingly the hitting set H'. If $g \in L(T_i)$, then $\mathsf{size}(g) \le O(dn)$. The D (and hence $R[\boldsymbol{x}]$) remains as before. Claims 4.7-4.8 hold similarly. We will construct the hitting set H' by showing that F has a small $\Sigma\wedge\Sigma\wedge$ circuit over $R[\boldsymbol{x}]$.

Note that, Theorem 4.10 remains the same for $\Sigma\wedge\Sigma\wedge$ (implying the same size blowup). However, Theorem 4.12, the size blowup is $O(d\,\mathsf{size}(g))$, because each monomial x^e can only produce $d+1$ many monomials. Therefore, similar to Theorem 4.15, one can show that $\Psi(J_{\boldsymbol{x}_k}(g_1,\ldots,g_k)) \in \Sigma\wedge\Sigma\wedge$, of size $O(2^k k!kds)$. Similarly, the size in Theorem 4.14 can be replaced by $s^{O(k)}$. Therefore, we get (similar to Theorem 4.9):

CLAIM 4.18. *$F \in R[\boldsymbol{x}]$ has $\Sigma\wedge\Sigma\wedge$ -circuit of size $s^{O(k)}$.*

Next, the degree bound also remains the same. Following the same footsteps, it is not hard to see that while degree bound on z_1 remains $\mathsf{poly}(ksd)$. Therefore, $P \in \mathbb{F}[\boldsymbol{x}, z_1]$ has $\Sigma\wedge\Sigma\wedge$ -circuit of size $s^{O(k)}$.

We want to *construct* a set $H' \subseteq \mathbb{F}^n$ such that the action $P(H', z_1) \neq 0$. By Theorem 2.9, we conclude that it has $s^{O(k\ \log\log s)}$ size hitting set which is constructible in a similar time. Hence, the construction of map Φ and the theorem follows (from z_1-degree bound).

Solution to Theorem 4.17. We know that

$$C(T_1,\ldots,T_m) = 0 \iff E := \Phi(C(T_1,\ldots,T_m)) = 0.$$

Since, H' can be constructed in $s^{O(k\log\log s)}$ time, it is trivial to find hitting set for $E|_{H'}$ (which is just a $k+2$-variate polynomial with the aforementioned degree bounds). The final hitting set for E can be constructed in $s'^{O(k)} \cdot s^{O(k\log\log s)}$ time. ∎

5. Conclusion. This work introduces the powerful DiDI-technique and solves three open problems in PIT for depth-4 circuits, namely $\Sigma^{[k]}\Pi\Sigma\Pi^{[\delta]}$ (blackbox) and $\Sigma^{[k]}\Pi\Sigma\wedge$ (both whitebox and blackbox). Here are some immediate questions of interest which require rigorous investigation.

1. Can the exponent in Theorem 1.1 be improved to $O(k)$? Currently, it is exponential in k.
2. Can we improve Theorem 1.3(b) to $s^{O(\log\log s)}$ (like in Theorem 1.3(a))?
3. Can we design a polynomial-time PIT for $\Sigma^{[k]}\Pi\Sigma\Pi^{[\delta]}$?
4. Design a polynomial time PIT for $\Sigma\wedge\Sigma\Pi^{[\delta]}$ circuits (i.e. unbounded top-fanin)?
5. Can we solve PIT for $\Sigma^{[k]}\Pi\Sigma\wedge^{[2]}$ efficiently (polynomial/quasipolynomial-time)?
6. Can we design an efficient PIT for rational functions of the form $\Sigma\,(1/\Sigma\wedge\Sigma)$ or $\Sigma\,(1/\Sigma\Pi)$ (for *un*bounded top-fanin)?

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