

Special-case Algorithms for Blackbox Radical Membership, Nullstellensatz and Transcendence Degree

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ABSTRACT

Radical membership testing, resp. its special case of Hilbert’s Nullstellensatz (HN), is a fundamental computational algebra problem. It is NP-hard; and has a famous PSPACE algorithm due to *effective* Nullstellensatz bounds. We identify a useful case of these problems where practical algorithms, & improved bounds, could be given—When transcendence degree (tr.deg) r of the input polynomials is smaller than the number of variables n . If d is the degree bound on the input polynomials, then we solve radical membership (even if input polynomials are *blackboxes*) in around d^r time. The prior best was $> d^n$ time (always, $d^n \geq d^r$). Also, we significantly improve effective Nullstellensatz degree-bound, when $r \ll n$.

Structurally, our proof shows that these problems reduce to the case of $r + 1$ polynomials of $\text{tr.deg} \geq r$. This input instance (corresponding to none or a *unique annihilator*) is at the core of HN’s hardness. Our proof methods invoke basic algebraic-geometry.

CCS CONCEPTS

• **Theory of computation** → **Algebraic complexity theory; Circuit complexity**; Problems, reductions and completeness.

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1 INTRODUCTION

Given a set of polynomials f_1, \dots, f_n , there is a natural certificate for the existence of a common root, namely the root itself. Hilbert’s Nullstellensatz [Rab30, Zar47, Kru50] states that there is also a natural certificate for the nonexistence of a common root, when the underlying field is *algebraically closed*. Formally, the theorem states that the polynomials have no common root if and only if there exist polynomials g_1, \dots, g_n such that $1 = \sum f_i g_i$. We refer to the latter type of certificate as a *Nullstellensatz certificate*. These certificates are not polynomial sized: every common root can have exponential bit complexity, and every set of witness polynomials g_i can have exponential degrees. This problem

is naturally of computational interest, since the generality of the statement affords reductions from many problems of interest. *Effective* versions of the Nullstellensatz have been extensively studied [Jel05, KPS⁺01, KPS99, Som99, Som97, BS91, Kol88, Bro87], and they allow the decision problem of existence of common roots (called HN) to be solved in polynomial space. Koiran [Koi96] proved that under generalized Riemann Hypothesis, HN can be solved in AM [AB09, Ch.8], for fields of characteristic zero.

In this work, we relate the complexity of HN to the transcendence degree of the input polynomials. The *transcendence degree* (tr.deg) of polynomials f_1, \dots, f_m is defined as the size of any maximal subset of the polynomials that are algebraically independent. This notion is well defined since algebraic independence satisfies matroid properties [Oxl06]. We show that HN can be solved in time single-exponential in tr.deg. This can be seen as a generalization of the fact that HN can be solved in time exponential in the number of polynomials (or variables) in the system. We state our result in terms of the question of *radical membership*: $f_0 \in ? \sqrt{\langle f_1, \dots, f_m \rangle}$. Note that the standard algorithms for both ideal membership [Her26] and radical computation [Lap06] are far slower than ours.

Given a set of polynomials f_1, \dots, f_m with tr.deg at most r , as blackboxes, we can perform radical membership tests for the ideal generated by f_1, \dots, f_m in time polynomial in d^r , m, n , where d is the degree-bound on the polynomials and n is the number of variables.

We also relate the tr.deg of the input polynomials to the degrees of the Nullstellensatz certificates, that is the degrees of g_i in $\sum f_i g_i = 1$; improving the best bounds by [Jel05].

Given a set of polynomials f_1, \dots, f_m with tr.deg r and without any common roots, there exist polynomials g_i of degree at most d^{r+1} such that $\sum f_i g_i = 1$.

We also give an output-sensitive algorithm to compute the tr.deg of polynomials. Slightly more formally, we show:

Given a set of polynomials f_1, \dots, f_m , we can compute their tr.deg in time polynomial in d^r and m, n .

1.1 Previously known results

All three of the problems stated above have been extensively studied. We therefore only list some of the previously known results, and direct readers to the surveys [May97, BS91].

Nullstellensatz. The decidability of the ideal membership problem was established by Hermann [Her26] when she proved a doubly-exponential bound on witnesses to ideal membership. A lower bound of the same complexity by Mayr and Meyer [May89, MM82] showed that this problem is EXPSpace complete. A number of different algorithms were developed for operations on ideals, most prominently the method of Gröbner basis [Buc65]. The proof of single-exponential bounds for the Nullstellensatz (discussed below) allowed special cases of the ideal membership problem, such

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as the case of unmixed and zero dimensional ideals to be solved in single-exponential time [DFGS91]. It also allowed the general Nullstellensatz problem to be solved in PSPACE. Giusti and Heintz [GH94] proved that the dimension of a variety can be computed by a randomized algorithm in single-exponential time, with the exponent being linear in n , which gives an algorithm of the same complexity for HN (by testing if the dimension is -1). All of the above results are independent of the underlying field characteristic. In 1996, Koiran [Koi96] gave an AM protocol (conditioned on GRH) for the Nullstellensatz problem, when the underlying field is \mathbb{C} and the polynomials have integer coefficients. His method is completely different from the previous methods (of using the effective Nullstellensatz to reduce the system to a linear one). The positive characteristic case is an open problem, and the best known complexity remains PSPACE.

Effective Nullstellensatz. The projective version of the effective Nullstellensatz follows from the fundamental theorem of elimination theory [Laz77]. An affine version was first proved by Brownawell [Bro87] in characteristic 0 using analytic methods. It was later improved by Kollár [Kol88] who used local cohomology to improve the bounds and remove the condition on the characteristic. A more elementary proof that used bounds on the Hilbert function was given by Sombra [Som97], who also gave improved bounds based on some geometric properties of related varieties [Som99]. An even more elementary and significantly shorter proof was given by Jelonek [Jel05], who obtained improved bounds when the number of polynomials is lesser than the number of variables.

Transcendence degree. Algebraic independence was studied in computer science by [DGW07] in their study of explicit extractors. They proved that the rank of the Jacobian matrix is the same as the tr.deg for fields of characteristic zero (or large enough) which gives an efficient randomized method for computing the tr.deg. The problem was studied further in [Kay09], where the condition on the characteristic for the above algorithm was relaxed, and some hardness results were established. [GSS18] showed that the problem is in $\text{coAM} \cap \text{AM}$, making it unlikely to be NP-hard, and conjecturing that the problem is in coRP for all characteristics. Algorithmically, the best known method for computing the tr.deg in fields of positive characteristic still has PSPACE complexity, by using the bounds of Perron [Per51, P105] to reduce the problem to solving an exponential sized linear system. This method takes time polynomial in d^{r^2} using the methods of [Csa75]. We refer the reader to the thesis [Sin19] for an exhaustive survey of related results; and applications in [ASSS12, PSS16].

Certain radical membership methods were developed by Gupta [Gup14] in his work on deterministic polynomial identity testing algorithms for heavily restricted depth-four circuits. The focus there however was on a *deterministic algorithm* for the above problem. Further, he restricts his attention to systems where the underlying field is \mathbb{C} .

1.2 Our results

Our algorithms will be Monte Carlo algorithms. We assume that our base field k is algebraically closed, but our algorithms only use operations in the field in which the coefficients of the inputs lie, which we denote by k_i . For example, k_i might be \mathbb{F}_p , and k

would then be $\overline{\mathbb{F}_p}$. By time complexity we mean operations in k_i , where operations include arithmetic operations, finding roots, and computing GCD of polynomials. Our results are valid for any field where the above procedures are efficient, for example finite fields.

We relate the complexity of radical membership, and the degree bounds in effective Nullstellensatz, to the tr.deg of the input set of polynomials. We do this by showing that given a system of polynomials, we can reduce both the number of variables and the number of polynomials to one more than the tr.deg, while preserving the existence (resp. non-existence) of common roots. In particular, when the tr.deg of the input polynomials is constant, we get efficient algorithms for these problems.

THEOREM 1.1 (RADICAL MEMBERSHIP). *Suppose f_1, \dots, f_m and g are polynomials, in variables x_1, \dots, x_n , of degrees d_1, \dots, d_m and d_g respectively, given as blackboxes. Suppose that $\text{tr.deg}(f_1, \dots, f_m) \leq r$. Define $d := \max(\max_i d_i, d_g)$.*

Then, testing if g belongs to the radical of the ideal generated by f_1, \dots, f_m can be done in time polynomial in n, m and d^r , with randomness.

Remarks:

(1) The tr.deg r can be much smaller than n , and this improves the complexity significantly to d^r from the prior d^n [LL91]. On the other hand, the usual reduction from SAT to HN results in a set of polynomials with transcendence degree n , due to the presence of polynomials $x_i^2 - x_i$ (that enforce the binary 0/1 values).

(2) We also show that the tr.deg itself can be computed in time d^r , independent of the characteristic (Theorem 1.3). In the above statement therefore, we can always pick $r = \text{tr.deg}(f)$, and we can assume that r is not part of the input.

(3) The tr.deg is upper bounded by the number of polynomials, and therefore we generalize the case of few polynomials. It is surprising if one contrasts this case with that of *ideal membership*—where the instance with three polynomials (i.e. tr.deg=3) is as *hard* as the general instance making it EXPSPACE-complete.¹

Next, we show that taking constant-free random linear combinations preserves the zeroset of the polynomials, if the number of linear combinations is at least one more than the tr.deg. This allows us to get bounds on the Nullstellensatz certificates that depend on the tr.deg.

THEOREM 1.2 (EFFECTIVE NULLSTELLENSATZ). *Suppose f_1, \dots, f_m are polynomials in x_1, \dots, x_n , of degrees $d_1 \geq \dots \geq d_m$ respectively, with an empty zeroset. Suppose further that $\text{tr.deg}(f_1, \dots, f_m) = r$.*

Then, there exist polynomials h_i such that $\deg f_i h_i \leq \prod_{i=1}^{r+1} d_i$ that satisfy $\sum f_i h_i = 1$.

Remark: The prior best degree-bound for the case of ‘small’ tr.deg is $\prod_{i=1}^m d_i$ [Jel05]. Our bound is significantly better when the tr.deg r is ‘smaller’ than the number of polynomials m .

Finally, as stated before, we show that the tr.deg of a given system of polynomials can be computed in time polynomial in d^r (and m, n), where d is the maximum degree of the input polynomials, and r is

¹ Suppose $g \in \langle f_1, \dots, f_m \rangle$ is an instance of ideal membership. This is equivalent to $z_1^m z_2^m g \in \langle z_1^{m+1}, z_2^{m+1}, \sum_i f_i z_1^i z_2^{m-i} \rangle$. Here, z_1, z_2 are fresh variables. This reduces the general instance of ideal membership to an instance where the ideal is generated by 3 elements. This transformation is from [Sap19].

their tr.deg. The algorithm is output-sensitive in the sense that the time-complexity depends on the output number r .

THEOREM 1.3 (TR.DEG). *Given as input polynomials f_1, \dots, f_m , in variables x_1, \dots, x_n , of degrees at most d , we can compute the tr.deg r of the polynomials in time polynomial in d^r, n, m .*

Remark: In the case when the characteristic of the field is greater than d^r , there is a much more efficient (namely, randomized polynomial time) algorithm using the Jacobian criterion [BMS13]. The algorithm presented here is useful when the characteristic is ‘small’; whereas the previous best known time-complexity was $> d^{r^2}$ if one directly implements the PSPACE algorithm. Eg. for $d = O(1)$ and $r = O(\log n)$ our complexity is polynomial-time unlike the prior known algorithms.

A motivating example where our results are better than the known results is when the input blackboxes are implicitly of the form $f_i(h_1, \dots, h_r), i \in [m]$, for $r \ll n$, where each h_i is an n -variate polynomial, and $m = n + 1$. Here, f_i ’s have transcendence degree r . Thus, our algorithms take time d^r ; significantly less than d^n .

1.3 Proof ideas

Pf. idea Theorem 1.1: We first use the Rabinowitsch trick to reduce to HN: the case $g = 1$. Next, we perform a random linear variable-reduction. We show that replacing each x_i with a linear combination of r new variables z_j preserves the existence of roots. This is done by using the fact that a general linear hyperplane intersects a variety properly (Lemma 3.1). Once we are able to reduce the variables, we can interpolate to get dense representation of our polynomials, and invoke existing results about testing nonemptiness of varieties (Theorem 2.6).

Pf. idea Theorem 1.2: For the second theorem, we show that random linear combinations of the input polynomials, as long as we take at least $r + 1$ many of them, preserve the zeroset. For this, we study the image of the polynomial map defined by the polynomials. We again use the theorem regarding the hyperplane intersection (Lemma 3.1). In order to get the degree bounds, we must allow these hyperplanes to depend on fewer variables, and allow their equations to be constant free. Once this is proved, we can use a bound (Theorem 2.5) on the Nullstellensatz certificates for the new polynomials (which is better since the polynomials are fewer in number) to obtain a bound for the original polynomials.

Pf. idea Theorem 1.3: The image of the polynomial map defined by the polynomials is such that the general fibre has codimension equal to the tr.deg. We first show that a random point, with coordinates from a subset which is not ‘too large’, satisfies this property. In order to efficiently compute the dimension of this fibre, we take intersections with hyperplanes; and apply Lemma 3.1 and Theorem 2.6.

2 NOTATION AND PRELIMINARIES

2.1 Notation

We reserve n for the number of variables (x_1, \dots, x_n) , m for the number of polynomials (f_1, \dots, f_m) in our inputs. The polynomials have total degrees d_1, \dots, d_m . We assume that the polynomials are labeled such that $d_1 \geq d_2 \geq \dots \geq d_m$.

We use boldface to denote sequence of objects, when the indexing set is clear; for example, \mathbf{x} denotes x_1, \dots, x_n and \mathbf{f} denotes f_1, \dots, f_m . The point $(0, \dots, 0)$ will be represented by $\mathbf{0}$. We use k to denote the underlying field which we assume is algebraically closed, and k_i to denote the field in which the coefficients of the inputs lie. We use \mathbb{A}^n to denote the n dimensional affine space over k . Given a variety X , we use $k[X]$ to denote its coordinate ring, and when X is irreducible we use $k(X)$ to denote its function field. We use \mathbb{A}^n and \mathbb{P}^n to denote the n dimensional affine and projective spaces respectively, and \mathbb{P}_∞^n to denote the hyperplane at infinity.

2.2 Algebraic-geometry facts

We use elementary facts from algebraic-geometry, for which [CLO07, SR13] are good references. We do not assume that our varieties (or zerosets) are irreducible. We will use the *Noether normalization lemma*. The following statement is useful, as it characterizes the linear maps which are Noether normalizing.

THEOREM 2.1. [SR13, Thm.1.15] *If $X \subseteq \mathbb{P}^N$ is a closed subvariety disjoint from an ℓ -dimensional linear subspace $E \subseteq \mathbb{P}^N$ then the projection $\pi : X \rightarrow \mathbb{P}^{N-\ell-1}$ with centre E defines a finite map $X \rightarrow \pi(X)$.*

Here, by *projection with center E* we mean that the coordinate functions of the map are the same as a set of defining linear equations for E . By the above theorem, proving that a given map is Noether normalizing for a particular variety reduces to proving that the variety is disjoint from a linear subspace.

We will also use the following two statements from dimension theory, namely the theorem on the dimension of intersections with hypersurfaces, and the theorem on the dimension of fibres.

THEOREM 2.2. [SR13, Thm.1.22] *If a form F is not zero on an irreducible projective variety X then $\dim(X \cap V(F)) = \dim X - 1$.*

THEOREM 2.3 (FIBRE DIMENSION). [SR13, Thm.1.25] *Let $f : X \rightarrow Y$ be a surjective regular map between irreducible varieties. Then $\dim Y \leq \dim X$, and for every $y \in Y$, the fibre $f^{-1}(y)$ satisfies $\dim f^{-1}(y) \geq \dim X - \dim Y$ (equiv. $\text{codim } f^{-1}(y) \leq \dim Y$).*

Further, there is a nonempty open subset $U \subset Y$: for every $y \in U$, $\dim f^{-1}(y) = \dim X - \dim Y$ (equiv. $\text{codim } f^{-1}(y) = \dim Y$).

The above theorem also holds if we replace surjective by *dominant*. Every fibre either is empty (if the point is not in the image) or has the above bound on the dimension. We sketch a proof of a special case of the above in Appendix A since we require an intermediate statement in the proof of Theorem 1.3.

We will also require the Bézout inequality. The definition of degree we use is the version more common in computational complexity. The degree of a variety is the sum of the degrees of all its irreducible components, as opposed to just the components of highest dimension. For irreducible varieties, the degree is the number of points when intersected with a general linear subspace of complementary dimension. This definition affords the following version of the *Bézout inequality* [Hei83], which holds without any conditions on the type of intersection.

THEOREM 2.4 (BÉZOUT [HEI83]). *Let X, Y be subvarieties of \mathbb{A}^n . Then $\deg(X \cap Y) \leq \deg X \cdot \deg Y$.*

Following is a recent version of *effective Nullstellensatz* [Jel05].

THEOREM 2.5. [Jel05, Thm.1.1] *Let f_1, \dots, f_m be nonconstant polynomials, from the ring $k[x_1, \dots, x_n]$ with k algebraically closed, that have no common zeros. Assume $\deg f_i = d_i$ with $d_1 \geq \dots \geq d_m$, and also $m \leq n$. Then, there exist polynomials h_i such that $\deg f_i h_i \leq \prod_{i=1}^m d_i$ satisfying $\sum f_i h_i = 1$.*

We will need the following algorithm for checking if a variety has dimension 0 (dim is an integer in the range $[-1, n]$). The statement assumes that the polynomials are given in the monomial (also called *dense*) representation. We only state the part of the theorem that we require. A discussion is provided in Appendix B. We note that the below theorem itself invokes results from [Laz81], section 8 of which proves that the operations occur in a field extension of degree at most d^n of the field k_i .

THEOREM 2.6. [LL91, Part of Thm.1] *Let f_1, \dots, f_m be polynomials of degree at most d in n variables. There exists a randomized algorithm that checks if the dimension of the zero set of f_1, \dots, f_m is 0 or not, in time polynomial in d^n, m . The error-probability is 2^{-d^n} .*

We will also require a bound on the degrees of *annihilators* of algebraically dependent polynomials. We refer to this bound as the Perron bound. It also plays a crucial role in the new proofs of effective Nullstellensatz (Theorem 2.5).

THEOREM 2.7 (PERRON BOUND). [BMS13, Cor.5] *Let f_1, \dots, f_m be algebraically dependent polynomials of degrees d_1, \dots, d_m . Then there exists a nonzero polynomial $A(y_1, \dots, y_m)$ of degree at most $\prod_{i=1}^m d_i$ such that $A(f_1, \dots, f_m)$ is identically zero.*

We note that the theorem statement in [BMS13] has the bound as $(\max d_i)^m$, however their method of constructing a linear faithful homomorphism and then applying the bound from [PI05] actually gives the above mentioned bound (even for the *weighted-degree* of A).

In the course of our proof, we will study the image of the polynomial map whose coordinate functions are f_1, \dots, f_m . We list some properties of this image.

LEMMA 2.8 (POLYNOMIAL MAP). *Let f_1, \dots, f_m be polynomials of degrees at most d , in variables x_1, \dots, x_n . Set $r := \text{tr.deg}(f_1, \dots, f_m)$. Let $F: \mathbb{A}^n \rightarrow \mathbb{A}^m$ be a polynomial map defined as*

$$F(a_1, \dots, a_n) = (f_1(a_1, \dots, a_n), \dots, f_m(a_1, \dots, a_n)).$$

Let Y be the (Zariski) closure of the image of \mathbb{A}^n under F , that is $Y := \overline{F(\mathbb{A}^n)}$. Then,

- (1) Y is irreducible.
- (2) $\dim Y = r$.
- (3) $\deg Y \leq d^r$.

PROOF OF LEMMA 2.8. The first statement is a consequence of the fact that Y is the image of an irreducible set (namely \mathbb{A}^n) under a continuous map. Since $k[Y] = k[f_1, \dots, f_m]$, we have $\text{tr.deg}(k(Y)) = r$, whence $\dim Y = r$ by definition. Here we used the fact that the dimension of an irreducible variety is the transcendence degree of its function field over the ground field. A proof of the third part can be found in [BCS97, 8.48]. \square

3 MAIN RESULTS

We require a bound on the probability that a random linear hyperplane intersects a variety of a given dimension properly, that is

such that the dimension of the variety decreases by exactly one. It is well known that the set of such hyperplanes form a Zariski open set in the space of all hyperplanes. We use an explicit bound on the probability of such an intersection based on the degree of the variety, both for the projective and the affine case. We will require that our intersecting hyperplanes have some structure: that their defining equations depend only on a few variables, depending on the dimension of the variety to be intersected. We establish all these facts in the next subsection. In the three subsections following that, we use this lemma to prove our three main results– Theorem 1.1, Theorem 1.2, and Theorem 1.3.

3.1 Intersection by a hyperplane

LEMMA 3.1. *Let $V \subseteq \mathbb{P}^n$ be a projective variety of dimension r and degree D . Let S be a finite subset, of the underlying field k , not containing 0. Let ℓ be a linear form in x_0, x_1, \dots, x_{n-r} with each coefficient picked uniformly and independently from S . Let H be the hyperplane defined by ℓ . Then, with probability at least $1 - D/|S|$ we have $\dim V \cap H = \dim V - 1$.*

Analogously, if $V \subseteq \mathbb{A}^n$ is affine, ℓ is a linear polynomial in x_1, \dots, x_{n-r+1} and H its hyperplane; then $\dim V \cap H = \dim V - 1$ with probability at least $1 - 2D/|S|$.

PROOF OF LEMMA 3.1. First we prove the projective case. Let $\ell := c_0 x_0 + \dots + c_{n-r} x_{n-r}$, where the c_i are the coefficients picked uniformly at random from S . Let $\cup_{j=1}^d V_j$ be the decomposition of the dimension- r part of V into irreducible components. Then by definition, $\deg V \geq \sum \deg V_j$, and hence $d \leq D$. Pick a point p_j in V_j , for each j . We can always pick p_j so that not all of its first $n - r + 1$ coordinates are zero: if this was not possible then V_j would have to be contained in the variety defined by $x_0 = x_1 = \dots = x_{n-r} = 0$, which has dimension only $r - 1$. By Theorem 2.2, $\dim H \cap \dim V_j = \dim V_j$ if and only if $V_j \subseteq H$ (since V_j and H are irreducible), and otherwise $\dim H \cap V_j = \dim V_j - 1$. The probability that this happens is upper bounded by the probability that $p_j \in H$. For a fixed j , this is equivalent to $\ell(p_j) = 0$. Since not all of the first $n - r + 1$ coordinates of p_j are zero, the above is bounded by $1/|S|$, by fixing all but one of the coordinates. By a union bound, with probability at most $d/|S|$, there exists some j where $\dim H \cap V_j = \dim V_j$. Therefore, with probability at least $1 - D/|S|$, we get $\dim V_j \cap H = \dim V_j - 1$ for every j , whence $\dim V \cap H = \dim V - 1$.

Now suppose V is affine. The difference from the projective case is that the intersection $V \cap H$ might be empty, and we need to bound the probability of this event. Let V^P be its projective closure. Then $\dim V^P = \dim V$ and $\deg V^P = \deg V$. By the previous part, we have $\dim V^P \cap H^P = \dim V^P - 1$ with probability $1 - D/|S|$. Then, the case $V \cap H = \emptyset$ only happens if $\dim V^P \cap H^P \cap \mathbb{P}_\infty^n = \dim V^P - 1$, where \mathbb{P}_∞^n is the hyperplane $x_0 = 0$ in \mathbb{P}^n . The irreducible components of V are in bijection with those of V^P , and hence V^P has no irreducible component contained in \mathbb{P}_∞^n . Therefore, $\dim V^P \cap \mathbb{P}_\infty^n = \dim V^P - 1$. Further, by Bézout's theorem we have $\deg V^P \cap \mathbb{P}_\infty^n \leq \deg V^P$.

Now $H^P \cap \mathbb{P}_\infty^n$ is a hyperplane in \mathbb{P}_∞^n defined by the nonconstant part of ℓ . In particular, it is a hyperplane whose defining equation has coefficients picked uniformly and independently and we can apply the projective version of this lemma on \mathbb{P}_∞^n . Therefore the probability that its intersection with $V^P \cap \mathbb{P}_\infty^n$ does not result in a reduction in the dimension is at most $D/|S|$. By a union bound, with

probability at least $1 - 2D/|S|$ it holds that $\dim V^P \cap H^P = \dim V - 1$ and $\dim V^P \cap H^P \cap \mathbb{P}_\infty^n = \dim V - 2$, whence $\dim V \cap H = \dim V - 1$ as required. \square

An important fact to note is that our choice of variables for the linear form is arbitrary. The lemma works for any choice of $n - r + 1$ variables, and this will be important when we use the lemma. Also, note that the above lemma works when our linear form involves more than $n - r + 1$ variables.

Repeated applications of the above allow us: (1) to reduce a variety to dimension 0 by taking hyperplane sections, and (2) to find a linear subspace that avoids the variety.

3.2 Radical membership: Proof of Thm.1.1

Using the above lemma, we complete the proof of the main theorem:

PROOF OF THEOREM 1.1. We first assume $g = 1$, which is the Nullstellensatz problem HN. Define $D := \prod_{i=1}^m d_i$, and $V := V(\langle \mathbf{f} \rangle)$. The set of common zeroes of these polynomials is the fibre of the point $\mathbf{0}$ under the map F defined in Lemma 2.8. The problem HN is thus equivalent to testing if a particular fibre of a polynomial map is nonempty. By the fibre dimension theorem (Theorem 2.3), the codimension of the zerset—if it is nonempty—is bounded above by the dimension of the image of the map, which by Lemma 2.8 is r . The zerset V is therefore either empty, or has dimension at least $n - r$. Assume that V is nonempty. By repeated applications of Bézout's theorem (Theorem 2.4), $\deg V \leq D$. Let S be a subset of the underlying field k_i (or an extension) of size at least $6(n - r)D$ that does not contain 0. We can sample from S in time polynomial in d, n, m , since S has size exponential in these parameters. Further, if we were required to go to an extension to form S , the degree of the extension would be polynomial in d, n, m . Pick $n - r$ random linear polynomials $\ell_1, \dots, \ell_{n-r}$ with coefficients from S , and call their zero sets H_1, \dots, H_{n-r} respectively. By Lemma 3.1, the intersection $V \cap H_1$ has dimension $r - 1$ with probability at least $1 - 1/(3(n - r))$. Further, by Bézout's theorem we get $\deg V \cap H_1 \leq \deg V \leq D$, since each H_i has degree one. Again by Lemma 3.1, the intersection $(V \cap H_1) \cap H_2$ has dimension $r - 2$ with probability at least $1 - 1/(3(n - r))$, and $\deg V \cap H_1 \cap H_2 \leq D$. Repeating this for all H_i and using the union bound, we get $\dim V \cap H_1 \cap \dots \cap H_{n-r} \geq 0$ with probability at least $2/3$.

Therefore, when the polynomials \mathbf{f} have nonempty zerset and are restricted to the r dimensional affine subspace $\cap H_i$, the new zerset has dimension at least 0, and in particular is nonempty. If the zerset of the polynomials was empty to begin with, then the restriction to the linear subspace also results in an empty zerset.

This restriction can be performed by a variable reduction, as follows. Treating \mathbb{A}^n as a vector space of dimension n over k , let H_0 be the linear subspace corresponding to the affine subspace $H := \cap H_i$. H_0 has dimension r , and hence has basis a_1, \dots, a_r . Further, let vector b be such that $H = H_0 + b$. Define linear forms c_1, \dots, c_n in new variables z_1, \dots, z_r as $c_i := \sum_{j=1}^r a_{ji} z_j + b_i$, where a_{ji} is the i^{th} component of a_j . Define $f'_i := f_i(c_1, \dots, c_n)$. Then by construction, the zerset of f'_1, \dots, f'_m is equal to $V \cap (\cap H_i)$. Further, $\deg f'_i = \deg f_i$, and these polynomials are in r variables. Also, the construction of these f'_i can be done in a blackbox manner,

given blackboxes for f_i . This construction takes time polynomial in m, r, n .

We now repeatedly invoke Theorem 2.6 to check if f'_i 's have a common root. First we must convert them to a sparse representation. The polynomial f'_i has at most $\binom{r+d_i}{r}$ many monomials, and therefore we can find every coefficient in time polynomial in $\binom{r+d_i}{r}$ by simply solving a linear system. Applying Theorem 2.6, we can test whether the dimension of the zerset of f'_1, \dots, f'_m is 0 or not. However, we want to check if the dimension is at least 0. For this, we randomly sample r more hyperplanes H'_1, \dots, H'_r as in the previous part of the proof, this time in the new variables z_1, \dots, z_r . Let V' be the zerset of f'_1, \dots, f'_m . We first use Theorem 2.6 to check if V' has dimension 0. If not, then we check if $V' \cap H'_1$ has dimension 0. If not, then we check $V' \cap H'_1 \cap H'_2$, and so on. We return success if any one of the above iterations returns success (implying that the corresponding variety has dimension 0). Performing calculations similar to the ones earlier in the proof, we see that with high probability each intersection reduces the dimension by 1. If V' originally had dimension r' , then after intersecting with r' hyperplanes, the algorithm of Theorem 2.6 returns success. If V' was empty, then the algorithm does not return success in any of the above iterations. This allows us to decide if V' has dimension at least 0. Finally, using the fact that the dimension of the zerset of f'_1, \dots, f'_m is at least 0 if and only if $\dim V \geq 0$, we get the required algorithm for HN.

We now estimate the time taken. Computing the dense representation takes time polynomial in d^r and m . Each of the at most r applications of Theorem 2.6 also take the same amount of time. The sampling steps take time polynomial in $\log nD$ (in turn polynomial in d, m) and only requires an extension of degree polynomial in n and $\log d$. The total time taken is therefore polynomial in m, d^r .

Now assume that g is an arbitrary polynomial. We reduce the problem to the case of $g = 1$ using Rabinowitsch trick [Rab30]. The polynomial g belongs to the radical of the ideal $\langle \mathbf{f} \rangle$ if and only if the polynomials $\mathbf{f}, 1 - yg$ have no common root (here y is a new variable). Further, if \mathbf{f} has transcendence degree r , then the set $\mathbf{f}, 1 - yg$ has transcendence degree $r + 1$. We therefore reduce the radical membership problem to HN problem, with a constant increase in the transcendence degree, number of polynomials and the number of variables. By the result in the previous paragraph, we can solve this in time polynomial in n, m and d^r . \square

3.3 Effective Nullstellensatz: Proof of Thm.1.2

We now prove that by taking random linear combinations of the input polynomials, we can reduce the number of polynomials to be one more than the transcendence degree while preserving the existence of roots. This reduction gives degree bounds for the Nullstellensatz certificates. Note that this reduction does not help in Section 3.2's root-testing procedure, since we will only be saving a factor in m if we reduce the number of polynomials.

THEOREM 3.2 (GENERATOR REDUCTION). *Let f_1, \dots, f_m be polynomials, in x_1, \dots, x_n , of degrees at most d and of $\text{tr.deg} = r$. Let g_1, \dots, g_{r+1} be polynomials defined as $g_i := \sum_{j=1}^m c_{ij} f_j$, where each c_{ij} is randomly picked from a finite subset S of k . Then with probability at least $1 - d^{(r+1)m}/|S|$, we have $V(\langle \mathbf{f} \rangle) = V(\langle \mathbf{g} \rangle)$.*

That we pick the linear combinations so that the first involves all polynomials, the second involves all except f_1 , the third involves all except f_1, f_2 and so on is crucial for the improvement in the degree bounds.

PROOF OF THEOREM 3.2. We prove this by studying the set Y defined in Lemma 2.8. Let $F : \mathbb{A}^n \rightarrow \mathbb{A}^m$ be the map with coordinate functions f_i . Let $Y := \overline{F(\mathbb{A}^n)}$, the closure of the image of F in \mathbb{A}^m . We use y_1, \dots, y_m to denote the coordinate functions of \mathbb{A}^m . By Lemma 2.8, Y has dimension r , and degree at most $D := d^r$. Let Y^P be the projective closure of Y . Then Y^P also has dimension r and degree at most D . Let $\ell_1, \dots, \ell_{r+1}$ be the linear polynomials $\ell_i := \sum_{i \leq j \leq m} c_{ij} y_j$.

Consider the subspace defined by $y_0, \ell_1, \dots, \ell_r$ in \mathbb{P}^m . The variety $Y^P \cap \mathbb{P}_{\infty}^m$, which is the intersection of Y^P with the hyperplane at infinity defined by $y_0 = 0$, has dimension $r - 1$. Since ℓ_1, \dots, ℓ_r are random linear polynomials and $Y^P \cap \mathbb{P}_{\infty}^m$ is a variety of, degree at most D and, dimension $r - 1$, we can repeatedly apply Lemma 3.1 to get a bound on the probability of proper intersections. Let H_i be the hyperplane defined by ℓ_i . We apply Lemma 3.1 starting from H_r . The equation ℓ_r has $m - r + 1$ coefficients, and therefore satisfies the conditions required for the lemma. By Bézout's theorem, the intersection has degree bounded by D , and dimension decreased by one. We then apply the theorem with H_{r-1} and so on, as in the proof of Theorem 1.1. In each iteration the variety considered has one less dimension than the previous iteration, but our linear polynomial has one more variable, and therefore we will always satisfy the conditions of Lemma 3.1.

We can now invoke Theorem 2.1 to say that the map $\mathbb{P}^m \rightarrow \mathbb{P}^r$ with coordinate functions $(y_0, \ell_1, \dots, \ell_r)$ is Noether normalizing for Y^P . We call this map L' . We use z_0, \dots, z_r to denote the coordinate functions of \mathbb{P}^r . The map L' sends the affine chart $y_0 \neq 0$ to the affine chart $z_0 \neq 0$. Let L be the restriction of L' to this affine chart. Then L defines a map from \mathbb{A}^m to \mathbb{A}^r , which is Noether normalizing for the variety Y ; we also call this restricted map L . More explicitly, the map L has coordinate functions (ℓ_1, \dots, ℓ_r) . Also, let the map $\mathbb{A}^m \rightarrow \mathbb{A}^{r+1}$ with coordinate functions $(\ell_1, \dots, \ell_{r+1})$ be labelled M .

Since the map L is Noether normalizing, it has finite fibres. Let Q be the fibre of $\mathbf{0}$ in Y . We bound the size of this set. The map L is Noether normalizing, and hence it is surjective. The image \mathbb{A}^r is normal, and hence the cardinality $|Q|$ of the fibre is bounded by the degree of the map [SR13, Theorem 2.28]. Here, by the degree of the map we mean the degree of $k(Y)$ over the pullback $L^*(k(\mathbb{A}^r))$. Note that $k(Y) = k(f_1, \dots, f_m)$, and $L^*(k(\mathbb{A}^r)) = k(\ell_1(\mathbf{f}), \dots, \ell_r(\mathbf{f}))$ after applying the same isomorphism. By Perron's bound, for each i there exists an annihilator of $f_i, \ell_1(\mathbf{f}), \dots, \ell_r(\mathbf{f})$ of degree at most d^{r+1} . The degree of the extension, and hence $|Q|$, is bounded by $d^{m(r+1)}$.

Further, no point of Q , other than $\mathbf{0}$, has all of the last $m - r$ coordinates as zero. This follows from the fact that $L^{-1}(\mathbf{0})$ is a linear space of dimension $m - r$, and its intersection with $y_{r+1} = y_{r+2} = \dots = y_m = 0$ has dimension 0. Consider now the linear form ℓ_{r+1} . For every $\mathbf{0} \neq q \in Q$, the probability that $\ell_{r+1}(q) = 0$ is at most $1/|S|$. Therefore, with probability at least $1 - d^{m(r+1)}/|S|$, the polynomial ℓ_{r+1} is nonzero on every nonzero point of Q .

Consider the polynomials g_1, \dots, g_{r+1} , and let G be the polynomial map $\mathbb{A}^n \rightarrow \mathbb{A}^{r+1}$ with coordinate functions g_i . By the choice of ℓ_i in the previous paragraph, the map G is exactly the composition of the map $F : \mathbb{A}^n \rightarrow \mathbb{A}^m$ with $M : \mathbb{A}^m \rightarrow \mathbb{A}^{r+1}$. Let Q be as defined earlier, the fibre of $\mathbf{0}$ under L . By construction, the set $M^{-1}(\mathbf{0})$ is a subset of Q . But since the polynomial ℓ_{r+1} is nonzero on every nonzero point of Q , the set $M^{-1}(\mathbf{0})$ consists only of $\mathbf{0}$. Therefore, $F^{-1}(M^{-1}(\mathbf{0})) = F^{-1}(\mathbf{0})$. Since $G = M \circ F$ we get $G^{-1}(\mathbf{0}) = F^{-1}(\mathbf{0})$; which is the same as $V(\langle \mathbf{f} \rangle) = V(\langle \mathbf{g} \rangle)$. \square

We use the above to prove our 2nd main result:

PROOF OF THEOREM 1.2. Using Theorem 3.2, there exists polynomials g_1, \dots, g_{r+1} of degrees d_1, \dots, d_{r+1} that do not have a common root. By Theorem 2.5, there exist h'_1, \dots, h'_{r+1} such that $\deg g_i h'_i \leq \prod_{i=1}^{r+1} d_i$ such that $\sum g_i h'_i = 1$. In this equation, substitute back the linear-combination of f_1, \dots, f_m for each g_i ; whence we get the required h_i 's. \square

3.4 Computing tr.deg: Proof of Thm.1.3

We give a method of 'efficiently' computing the tr.deg of input polynomials f_1, \dots, f_m . By Lemma 2.8 and the second part of Theorem 2.3, the tr.deg can be computed if we know the dimension of a general fibre. We need to get a bound on the points that violate the equality in Theorem 2.3. For this we follow the classical proof of the theorem and give effective bounds wherever required. For convenience we have provided a proof sketch in Appendix A, for the special case we need.

LEMMA 3.3. *Let h_1, \dots, h_m be polynomials of degree at most d in n variables, and let W be the Zariski closure of the image of the map \mathbf{h} with coordinates h_i . Let $S \subset k$ be of size $6nd^n$. If a_1, \dots, a_n are randomly picked from S , then with probability at least $5/6$, the fibre of $(h_1(\mathbf{a}), \dots, h_m(\mathbf{a}))$ has codimension exactly $\dim W$.*

PROOF. First assume that the h_i are algebraically independent. Then $W = \mathbb{A}^m$. Let the input variables be labelled such that $x_1, \dots, x_{n-m}, h_1, \dots, h_m$ are algebraically independent, and let $A_j(z_0, z_1, \dots, z_{n-m}, w_1, \dots, w_m)$ be the (minimal) annihilator of x_j over this set of variables, that is $A_j(x_j, x_1, \dots, x_{n-m}, h_1, \dots, h_m) = 0$. By the proof of Theorem 2.3 (Appendix A), a sufficient condition for point a_1, \dots, a_n to be such that $\mathbf{h}(\mathbf{a})$ has fibre of dimension exactly $n - m$ is that $A_j(x_j, x_1, \dots, x_{n-m}, h_1(\mathbf{a}), \dots, h_m(\mathbf{a}))$ is a nonzero polynomial. The polynomial A_j , when treated as polynomials in variables z_0, \dots, z_{n-m} with coefficients in $k[w_1, \dots, w_m]$ are such that the leading monomial has coefficient a polynomial in w_1, \dots, w_m of weighted-degree at most $\prod_{i=1}^m d_i$ (by Perron bound). By the polynomial identity lemma [Ore22, DL78, Sch80, Zip79], if we pick each a_i randomly from a set of size $6 \prod_{i=1}^m d_i$ then, with probability at least $5/6$, none of the polynomials $A_j(x_j, x_1, \dots, x_{n-m}, h_1(\mathbf{a}), \dots, h_m(\mathbf{a}))$ is zero. In this case, the codimension of the fibre of $\mathbf{h}(\mathbf{a})$ is exactly m as claimed.

In the general case, the h_i may be algebraically dependent, and W is a subvariety of \mathbb{A}^m . Suppose $\dim W = \text{tr.deg}(\mathbf{h}) =: s$. Then we take s many random linear combinations g_i of the h_i , as in the proof of Theorem 1.2. The map defined by the g_i is dense in \mathbb{A}^s and therefore the g_i ($i \in [s]$) are algebraically independent. By the previous paragraph, point \mathbf{a} picked coordinatewise from S is such that the fibre of $\mathbf{g}(\mathbf{a})$ has codimension s . The fibre of $\mathbf{h}(\mathbf{a})$ is

a subset of the fibre of $\mathbf{g}(\mathbf{a})$, and therefore it has codimension at least s . Finally, by Theorem 2.3, the fibre has codimension at most s , whence the fibre of $\mathbf{h}(\mathbf{a})$ has $\text{codim} = s$ as required. \square

PROOF OF THEOREM 1.3. For each i , upwards from 1 to n , we do the following steps. We iterate till i reaches $\text{tr.deg } r$ of the m polynomials. In the i -th iteration, we intersect \mathbb{A}^n with $n-i$ random hyperplanes $\ell_1, \dots, \ell_{n-i}$, as in the proof of Theorem 1.1 (Sec.3.2). Here, the coefficients are picked from a set S of size at least $n \cdot 18 \prod_{i=1}^m d_i$. We therefore reduce the problem to i variables.

Randomly pick point \mathbf{a} where each coordinate (of the n many) is picked randomly from S . By Lemma 3.3 (& 3.1), with error-probability $\leq 1/6n$, the point $\mathbf{f}(\mathbf{a})$ has intersected fibre of dimension $(n-r) - (n-i) = (i-r)$. We need to check this algorithmically; which is done by interpolating the polynomials \mathbf{f} after hyperplane intersections, and then using Theorem 2.6 (as detailed in Sec.3.2). If the intersected fibre dimension is zero, we have certified $\text{tr.deg} = i = r$; so we halt and return i as output. Else, we move to the next $i \mapsto i+1$. The interpolation step above is performed by solving a linear system which has size polynomial in d^i which is the count of the monomials of degree at most d in i variables.

Note that for $i < r$, with error-probability $\leq 1/6n$, the fibre of $\mathbf{f}(\mathbf{a})$ has an empty intersection with $\ell_1, \dots, \ell_{n-i}$; which is $\text{dim} = -1$ and hence gets verified by Theorem 2.6.

By a union bound therefore, with error-probability $\leq 1/6$, the above algorithm gives the correct answer. For each i , the time complexity of the above steps is polynomial in d^i, m , which is the time taken for the interpolation step and to verify zero-dimension of the fibre. Therefore the algorithm as a whole takes time polynomial in d^r, n, m as claimed. \square

4 CONCLUSION

We give algorithms for radical membership and tr.deg of systems of polynomials, in time that depends on the tr.deg . In both cases, our algorithms generalize the cases of ‘few’ input polynomials. We further give bounds on the degree of the Nullstellensatz certificates that depend on the tr.deg of the input polynomials. In all three cases, our bounds are significantly better than the previously known results in the regime when the tr.deg is much smaller than the number of variables and the number of polynomials.

Our work leaves the natural open problem of designing efficient algorithms when the tr.deg is ‘larger’.

- For the blackbox radical membership problem, given the NP-hardness of HN, it is unlikely that a significantly better algorithm exists (unless other restrictions are put on the input polynomials).
- Could our methods, and the core hard instance thus identified, help in proving that HN is in AM? Currently, this is known only partially [Koi96].
- For the tr.deg problem however, we know that the problem is in $\text{coAM} \cap \text{AM}$, making it unlikely to be NP hard. It is therefore likely that there is an efficient randomized algorithm whose time complexity is polynomial in n and m . This is already known in the case when the field has large/zero characteristic, and it is an open problem to extend this to other fields. A first step might be to give a subexponential

time algorithm for the problem that works without any assumptions.

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