<u>Defn 1</u>: An <u>ABP</u> is a layered graph with unique <u>source</u> (resp. <u>sink</u>) vertex s (resp. t). Edges from layers i to $(i+1)$ are labelled by a linear poly.

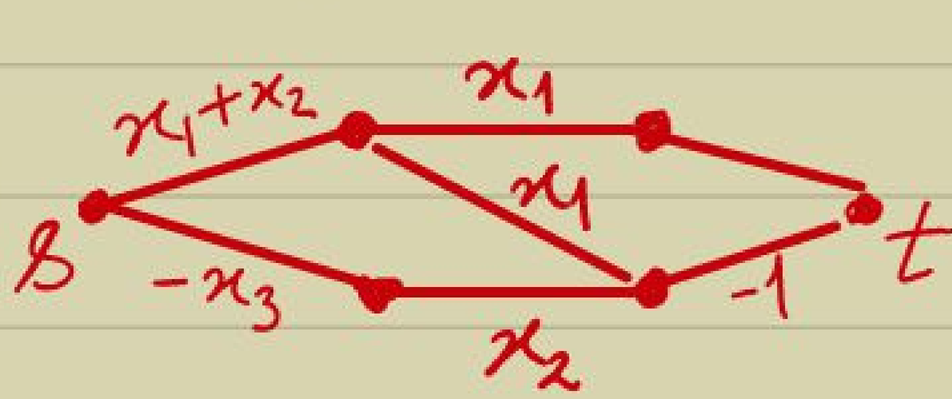
The <u>polynomial</u> computed by the ABP is $f = \sum_{\text{path } \gamma : s \rightsquigarrow t} wt(\gamma)$,

where <u>$wt(\gamma)$</u> is the product of the edge weights in γ.

The <u>width</u> of the ABP is the max number of vertices in any layer.

The <u>depth</u> is the length of the max path from $s \rightsquigarrow t$.

– Eg.



computes $f = x_2 x_3$

in width $=2$ & depth $=3$.

- In this example, note that we can also represent f by using the adjacency matrices of the level transitions.

E.g. $[x_1+x_2, -x_3]\begin{bmatrix} x_1 & x_1 \\ 0 & x_2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = x_2 x_3.$

Also $= [1, -1]\begin{bmatrix} x_1+x_2 & 0 \\ 0 & x_3 \end{bmatrix}\begin{bmatrix} x_1 & x_1 \\ 0 & x_2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}.$

Defn: Iterated matrix multiplication polynomial $IMM_{n,d}$ is the $(1,1)$-th entry of the product

$$X^{(1)} \cdot X^{(2)} \cdots X^{(d)},$$

where $X^{(i)}$ are $n \times n$ symbolic matrices (i.e. with each entry being a constant or a variable).

Theorem: If f has a width-w, depth-d ABP, then it has an IMM of size $O((wn)^2 \cdot d)$.

If f has an $IMM_{w,d}$ then it has a width-w, depth-d ABP.

Pf: (easy exercise)

▷ Any polynomial $f \in \mathbb{F}[x_1,\ldots,x_n]$ of sparsity s & degree d has an ABP of size $O(sd)$.

Pf: Build the ABP for each monomial. $\square$

Defn: Symbolic determinant $D(\bar{x})$ is a polynomial that equals the determinant of an $m\times m$ matrix with entries as $\mathbb{F} \cup \bar{x}$. (m is the size)

- Our next big connection is the one between ABP & symbolic determinant.

  For that, we need a graph interpretation of determinant.

- Any permutation $\sigma \in Sym(n)$ can be decomposed into cycles. Eg. $[5] \to (1,3,2,5,4)$ has the cycle decomposition: $(2\ 3)(4\ 5)$ with sign $= (-1)^{\#\text{even-cycles}} = (-1)^2 = 1$.

- In a graph G, a cycle cover is a

partition of $V(G)$ into cycles (simple, disjoint).

Theorem: Let G be the graph on $V(G) = [n]$ with adjacency matrix $X = (x_{ij})_{n \times n}$. Then,

$$\det(X) = \sum_{C \in \mathrm{CycleCover}(G)} \mathrm{sgn}(C) \cdot \mathrm{wt}(C),$$

where $\mathrm{sgn}(C) := (-1)^{\#\text{even-cycles in } C}$ &

$$\mathrm{wt}(C) := \prod_{\text{edge } e \in C} \mathrm{wt}(e).$$

Proof: • We have $\det(X) = \sum_{\sigma \in \mathrm{Sym}(n)} \mathrm{sgn}(\sigma) \cdot \prod_{i \in [n]} x_{i,\sigma(i)}$.

• The summand corresponding to σ can be seen to be equal to $\mathrm{sgn}(C) \cdot \mathrm{wt}(C)$, where C is the cycle cover of G specified by the cycle decomposition of σ.

• Finally, note that every cycle cover C of G uniquely specifies some $\sigma \in \mathrm{Sym}(n)$. $\square$

Corollary: $\mathrm{per}_n(X) = \sum_{C \in \mathrm{CycleCover}(G)} \mathrm{wt}(C)$.

- We are ready to reduce ABP to det.

Lemma: If f has a width-w, depth-d ABP, then it has a $O(wdn)$-size determinant.

Proof:

- We can first make the ABP edge weights symbolic, i.e. in $\mathbb{F} \cup \overline{x}$.

Ensure length d' paths $\rightarrow$ This makes the depth $O(d) =: d'$.

- Let G be the directed graph underlying this ABP. Modify it to a graph G':
  - Add a $\mathrm{wt}=1$ edge from t to s.
  - On all other vertices add a $\mathrm{wt}=1$ self-loop.
- Observe that any $C \in \mathrm{cycleCover}(G')$ uniquely specifies a path from $s \rightsquigarrow t$, & the $\mathrm{sgn}(C)$ is the same for all C (say 1).

- Thus, det of the adjacency matrix A' of G' is $\det(A') = f$.
- Clearly, A' has dimension $wnd' = O(wdn)$.

$\square$

- More surprising is the converse:

<u>Theorem</u> (Mahajan, Vinay '97): $\det_n$ has a width-$O(n^2)$, depth-$O(n)$ ABP, over any $\mathbb{F}$.

[$\Rightarrow \det_n \in VP$ (depth-$\lg n$) (P-uniform)]
(unbounded fanin/fanout)

- The main tool in the proof is a relaxation of disjoint cycles to <u>closed walks</u> (while still having the det connection).

<u>Defn</u>: Let G be a graph on $V(G) = [n]$.
A <u>clow</u> of G is a closed walk of length, say, ℓ such as $C = (v_1, v_2, \ldots, v_\ell, v_1)$ with v_1 being <u>unique min</u>. <u>head(C)</u> is v_1.

[head does not repeat in a clow.]

A clow sequence is a clow-tuple $(C_1, \ldots, C_r)$ with increasings heads, i.e. $head(C_1) < \cdots < head(C_r)$.

The length of a clow sequence is the sum of the lengths of the underlying clows.

fix it to be n

The weight of a clow sequence is the product of the weights of the underlying edges.

The sign of a clow sequence is $(-1)^{\#\text{even-clows}}$. [Even-clow has even length]

▷ A cycle cover is a clow sequence of the same weight & sign.

[Obviously, converse is false.]

- The surprise is:

Lemma [Mahajan, Vinay 1997]: If A is the adjacency matrix of G, then $\det(A) = \sum_{C \in ClowSequence(G)} sgn(C) \cdot wt(C)$.

Proof: • The key idea is to show that the contributions of clow sequences, that are not cycle covers, cancel each other!

• Consider a clow seq. $C = (C_1, \ldots, C_r)$ of length l. If C is not a cycle cover then some vertex must repeat.

• Let $i \in [r]$ be the largest such that $C_i = (v_1, v_2, \ldots, v_k, v_1)$ has a vertex that repeats (somewhere in $C_i, C_{i+1}, \ldots, C_r$).

$\Rightarrow (C_{i+1}, \ldots, C_r)$ are disjoint cycles but $(C_i, \ldots, C_r)$ are not.

• This can happen in two ways:

Case 1: $\exists j' < j \in [k]$, $v_{j'} = v_j$.

Case 2: $\exists j \in [k]$, v_j occurs in $C_{i+1}, \ldots,$ or C_r.

Over the cases, pick the least j.

This cycle is disjoint from $C_{i+1}, \dots, C_r$.

• In case 1, vertices $v_{j'+1}, \dots, v_j$ are all distinct (as v_j is the first occurrence of a repeated node). So, this gives a cycle.

• Define a new clow seq. C' by breaking C_i into the clow $(v_1, \dots, v_{j'}, v_{j+1}, \dots, v_k, v_1)$ & the cycle $(v_{j'+1}, \dots, v_j)$.

Note: The heads in C' are distinct.
else use Case 2

Note that $wt(C') = wt(C)$ & $sgn(C') = (-1)^{\ell+r+1} = -sgn(C)$.

$\# odds \equiv_2 \sum_{\text{even clow } C_i} |C_i| + \sum_{\text{odd } C_i} |C_i| = \ell = n$

• In case 2, $v_j \in C_i$ also appears in $C_{i'}$ for $i < i' \leq r$. (Note: i' is unique.)

Here we join the clows C_i & $C_{i'}$ at the vertex v_j to get the clow $C_i' :=$
$(v_1, \dots, v_j, C_{i'} \setminus \{v_j\}, v_{j+1}, \dots, v_k, v_1)$.

Note: $v_1 = head(C_i) < head(C_{i'})$

Call the new clow sequence C'.

• Note that $wt(C') = wt(C)$ & $sgn(C') = (-1)^{\ell+r-1} = -sgn(C)$.

▷ The above gives us a map τ from

ClowSequence(G) \ CycleCover(G) to itself such that: τ has <u>no</u> fixed point, it is invertible, & flips the sign.

- Thus, $\sum_{C \in \text{ClowSequence}(G)} \text{sgn}(C) \cdot \text{wt}(C)$ ------(1)

$$= \sum_{C \in \text{CycleCover}(G)} \text{sgn}(C) \cdot \text{wt}(C)$$

$$= \det(A). \qquad \square$$

- The sum over clow sequences can be computed by an ABP:

<u>Lemma</u> [Mahajan, Vinay '97] Expression (1) has a width-n^2, depth-$(n+1)$ ABP, where G has n vertices.

- ABP-width corresponds to memory (<u>registers</u>) & depth corresponds to <u>time</u>.

Proof: • The layers are labelled $l \in [n+1]$.

Layer $l \in [2 \ldots n]$ has $\Theta(n^2)$ nodes labelled $v_{i,j}^{(l)}$, $i \neq j \in [n]$.

Layer $l = 1$ has the node $s = v_{1,1}^{(1)}$ with $wt = x_{1i}$ edge to $v_{1,i}^{(2)}$, $\forall\, 1 < i \leq n$.

• Idea: In $v_{i,j}^{(l)}$, i remembers the head & j the current node in the current clow.

With this we intend to hard-code a clow sequence as a path $s \to t$ & vice versa.

For this the ABP has:

1) $\forall i < j \in [n], 1 < l < n$, $v_{i,j}^{(l)}$ has an edge of $wt = x_{jk}$ to $v_{i,k}^{(l+1)}$, for all $k > i$.

[grows the i-headed clow from j to k.]

2) $\forall i < j \in [n], 1 < l < n$, $v_{i,j}^{(l)}$ has an edge of $wt = -x_{ji}$ to $v_{k,k}^{(l+1)}$, for all $k > i$.

[clow ends, sign changes & new head=k.]

3) Last layer: $\forall i<j\in[n]$, $v_{i,j}^{(n)}$ has an edge of $wt = -x_{ji}$ to t.

sign in a path $= (-1)^{\#clows}$ [clow ends, sign changes & the clow sequence ends.]

▷ Each ABP path corresponds to a unique clow sequence of G. Moreover, the respective weight & signed-weight match.

with first-head = 1

▷ Each clow sequence of G corresponds to a unique path in ABP.

$\square$

Corollary: 1) $\det_n$ has an $IMM_{O(n^2), O(n)}$.

2) $\det_n$, over any commutative ring, has $O(\lg n)$-depth, unbounded fanin/out, poly(n)-size arithmetic circuit.

[det over noncommutative ring?]