Defn 1: An ABP is a layered graph with unique source (resp. sink) vertex s (resp. t). Edges from layers i to $(i+1)$ are labelled by a linear poly.

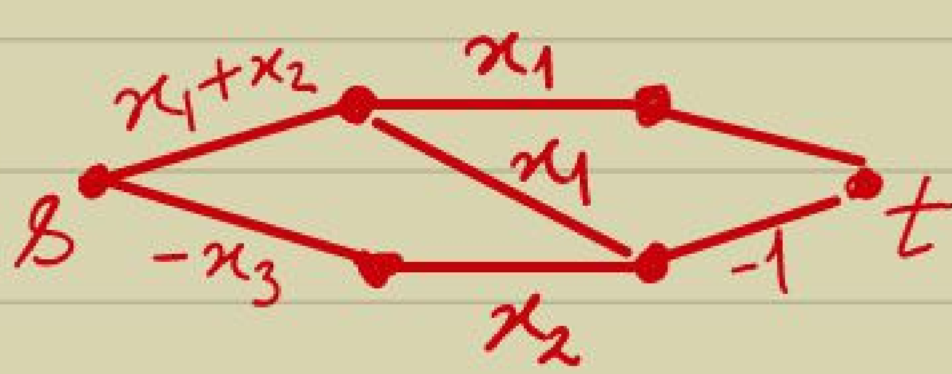
The polynomial computed by the ABP is $f = \sum_{\text{path } \gamma: s \rightsquigarrow t} wt(\gamma)$,

where $wt(\gamma)$ is the product of the edge weights in γ.

The width of the ABP is the max number of vertices in any layer.

The depth is the length of the max path from $s \rightsquigarrow t$.

- Eg.



computes $f = x_2 x_3$

in width $=2$ & depth $=3$.

- In this example, note that we can also represent f by using the adjacency matrices of the level transitions.

E.g. $[x_1+x_2, -x_3]\begin{bmatrix} x_1 & x_1 \\ 0 & x_2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = x_2x_3$.

Also $= [1, -1]\begin{bmatrix} x_1+x_2 & 0 \\ 0 & x_3 \end{bmatrix}\begin{bmatrix} x_1 & x_1 \\ 0 & x_2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Defn: Iterated matrix multiplication polynomial $IMM_{n,d}$ is the (1,1)-th entry of the product

$$X^{(1)} \cdot X^{(2)} \cdots X^{(d)},$$

where $X^{(i)}$ are $n \times n$ symbolic matrices (i.e. with each entry being a constant or a variable).

Theorem: If f has a width-w, depth-d ABP, then it has an IMM of size $O((wn)^2 \cdot d)$.

If f has an $IMM_{w,d}$ then it has a width-w, depth-d ABP.

Pf: (easy exercise)

▷ Any polynomial $f \in \mathbb{F}[x_1, \ldots, x_n]$ of sparsity s & degree d has an ABP of size $O(sd)$.

Pf: Build the ABP for each monomial. $\square$

Defn: Symbolic determinant $D(\bar{x})$ is a polynomial that equals the determinant of an $m \times m$ matrix with entries as $\mathbb{F} \cup \bar{x}$. (m is the size)

- Our next big connection is the one between ABP & Symbolic determinant.

For that, we need a graph interpretation of determinant.

- Any permutation $\sigma \in Sym(n)$ can be decomposed into cycles. Eg. $[5] \to (1,3,2,5,4)$ has the cycle decomposition: $(2\ 3)(4\ 5)$ with $sign = (-1)^{\#\text{even-cycles}} = (-1)^2 = 1$.

- In a graph G, a cycle cover is a

partition of $V(G)$ into cycles (simple, disjoint).

<u>Theorem</u>: Let G be the graph on $V(G) = [n]$ with adjacency matrix $X = (x_{ij})_{n \times n}$. Then,

$$\det(X) = \sum_{C \in \mathrm{cycleCover}(G)} \mathrm{sgn}(C) \cdot \mathrm{wt}(C),$$

where $\underline{\mathrm{sgn}(C)} := (-1)^{\#\text{even-cycles in } C}$ &

$$\underline{\mathrm{wt}(C)} := \prod_{\text{edge } e \in C} \mathrm{wt}(e).$$

<u>Proof</u>: • We have $\det(X) = \sum_{\sigma \in \mathrm{Sym}(n)} \mathrm{sgn}(\sigma) \cdot \prod_{i \in [n]} x_{i,\sigma(i)}$.

• The summand corresponding to σ can be seen to be equal to $\mathrm{sgn}(C) \cdot \mathrm{wt}(C)$, where C is the cycle cover of G specified by the cycle decomposition of σ.

• Finally, note that <u>every</u> cycle cover C of G uniquely specifies some $\sigma \in \mathrm{Sym}(n)$. $\square$

Corollary: $per_n(X) = \sum_{C \in CycleCover(G)} wt(C)$.

- We are ready to reduce ABP to det.

Lemma: If f has a width-w, depth-d ABP, then it has a $O(wdn)$-size determinant.

Proof:

- We can first make the ABP edge weights symbolic, i.e. in $\mathbb{F} \cup \overline{x}$.

Ensure length d' paths → This makes the depth $O(d) =: d'$.

- Let G be the directed graph underlying this ABP. Modify it to a graph G':
  - Add a $wt=1$ edge from t to s.
  - On all other vertices add a $wt=1$ self-loop.
- Observe that any $C \in cycleCover(G')$ uniquely specifies a path from $s \rightsquigarrow t$, & the $sgn(C)$ is the same for all C (say 1).

- Thus, det of the adjacency matrix A' of G' is $\det(A') = f$.
- Clearly, A' has dimension $wnd' = O(wdn)$.

$\square$

- More surprising is the converse:

Theorem (Mahajan, Vinay '97): $\det_n$ has a width-$O(n^2)$, depth-$O(n)$ ABP, over any $\mathbb{F}$.

[$\Rightarrow \det_n \in VP$ (depth-$\lg n$) (P-uniform)]
(unbounded fanin/fanout)

- The main tool in the proof is a relaxation of disjoint cycles to closed walks (while still having the det connection).

Defn: Let G be a graph on $V(G) = [n]$.
A clow of G is a closed walk of length, say, ℓ such as $C = (v_1, v_2, \ldots, v_\ell, v_1)$ with v_1 being unique min. head(C) is v_1.

[head does not repeat in a clow.]

A clow sequence is a clow-tuple $(C_1, \ldots, C_r)$ with increasings heads, i.e. $head(C_1) < \cdots < head(C_r)$.

fix it to be n → The length of a clow sequence is the sum of the lengths of the underlying clows.

The weight of a clow sequence is the product of the weights of the underlying edges.

The sign of a clow sequence is $(-1)^{\#\text{even-clows}}$. [Even-clow has even length]

▷ A cycle cover is a clow sequence of the same weight & sign.

[Obviously, converse is false.]

- The surprise is:

Lemma [Mahajan, Vinay 1997]: If A is the adjacency matrix of G, then $\det(A) = \sum_{C \in ClowSequence(G)} sgn(C) \cdot wt(C)$.

Proof: • The key idea is to show that the contributions of clow sequences, that are not cycle covers, cancel each other!

• Consider a clow seq. $C = (C_1, \ldots, C_r)$ of length l. If C is not a cycle cover then some vertex must repeat.

• Let $i \in [r]$ be the largest such that $C_i = (v_1, v_2, \ldots, v_k, v_1)$ has a vertex that repeats (somewhere in $C_i, C_{i+1}, \ldots, C_r$).

$\Rightarrow (C_{i+1}, \ldots, C_r)$ are disjoint cycles but $(C_i, \ldots, C_r)$ are not.

• This can happen in two ways:

Case 1: $\exists\, j' < j \in [k]$, $v_{j'} = v_j$.

Case 2: $\exists\, j \in [k]$, v_j occurs in $C_{i+1}, \ldots,$ or C_r.

Over the cases, pick the least j.

This cycle is disjoint from $C_{i+1}, \ldots, C_r$.

Note: The heads in C' are distinct.

else use Case 2

• In case 1, vertices $v_{j'+1}, \ldots, v_j$ are all distinct (as v_j is the first occurrence of a repeated node). So, this gives a cycle.

• Define a new clow seq. C' by breaking C_i into the clow $(v_1, \ldots, v_{j'}, v_{j+1}, \ldots, v_k, v_1)$ & the cycle $(v_{j'+1}, \ldots, v_j)$.

Note that $wt(C') = wt(C)$ &

$$sgn(C') = (-1)^{\ell+r+1} = -sgn(C).$$

$$\#\text{odds} \equiv_2 \sum_{\text{even clow } C_i} |C_i| + \sum_{\text{odd } C_i} |C_i| = \ell = n$$

• In case 2, $v_j \in C_i$ also appears in $C_{i'}$ for $i < i' \leq r$. (Note: i' is unique.)

Here we join the clows C_i & $C_{i'}$ at the vertex v_j to get the clow $C_i' :=$

$$(v_1, \ldots, v_j, C_{i'} \setminus \{v_j\}, v_{j+1}, \ldots, v_k, v_1).$$

Note: $v_1 = \text{head}(C_i) < \text{head}(C_{i'})$

Call the new clow sequence C'.

• Note that $wt(C') = wt(C)$ &

$$sgn(C') = (-1)^{\ell+r-1} = -sgn(C).$$

▷ The above gives us a map τ from

ClowSequence(G) \ CycleCover(G) to itself such that: τ has <u>no</u> fixed point, it is invertible, & flips the sign.

- Thus, $\sum_{C \in \text{ClowSequence}(G)} \text{sgn}(C) \cdot \text{wt}(C)$ -----(1)

$$= \sum_{C \in \text{CycleCover}(G)} \text{sgn}(C) \cdot \text{wt}(C)$$

$$= \det(A). \qquad \square$$

- The sum over clow sequences can be computed by an ABP:

<u>Lemma</u> [Mahajan, Vinay '97] Expression (1) has a width-n^2, depth-$(n+1)$ ABP, where G has n vertices.

- ABP-width corresponds to memory (<u>registers</u>) & depth corresponds to <u>time</u>.

Proof: • The layers are labelled $\ell \in [n+1]$.

Layer $\ell \in [2 \dots n]$ has $\Theta(n^2)$ nodes labelled $v_{i,j}^{(\ell)}$, $i \neq j \in [n]$.

Layer $\ell = 1$ has the node $s = v_{1,1}^{(1)}$ with $wt = x_{1i}$ edge to $v_{1,i}^{(2)}$, $\forall\, 1 < i \leq n$.

• Idea: In $v_{i,j}^{(\ell)}$, i remembers the head & j the current node in the current clow.

With this we intend to hard-code a clow sequence as a path $s \to t$ & vice versa.

For this the ABP has:

1) $\forall\, i < j \in [n], 1 < \ell < n$, $v_{i,j}^{(\ell)}$ has an edge of $wt = x_{jk}$ to $v_{i,k}^{(\ell+1)}$, for all $k > i$.

[grows the i-headed clow from j to k.]

2) $\forall\, i < j \in [n], 1 < \ell < n$, $v_{i,j}^{(\ell)}$ has an edge of $wt = -x_{ji}$ to $v_{k,i}^{(\ell+1)}$, for all $k > i$.

[clow ends, sign changes & new head $= k$.]

3) Last layer: $\forall i<j \in [n]$, $v_{i,j}^{(n)}$ has an edge of $wt = -x_{ji}$ to t.

Sign in a path $= (-1)^{\#\text{clows}}$ [clow ends, sign changes & the clow sequence ends.]

▷ Each ABP path corresponds to a unique clow sequence of G. Moreover, the respective weight & signed-weight match.

▷ Each clow sequence with first-head = 1 of G corresponds to a unique path in ABP.

$\square$

<u>Corollary</u>: 1) $\det_n$ has an $IMM_{O(n^2), O(n)}$.

2) $\det_n$, over any commutative ring, has $O(\lg n)$-depth, unbounded fanin/out, poly(n)-size arithmetic circuit.

[det over noncommutative ring?]