Defn 1: An ABP is a layered graph with unique source (resp. sink) vertex s (resp. t). Edges from layers i to $(i+1)$ are labelled by a linear poly.

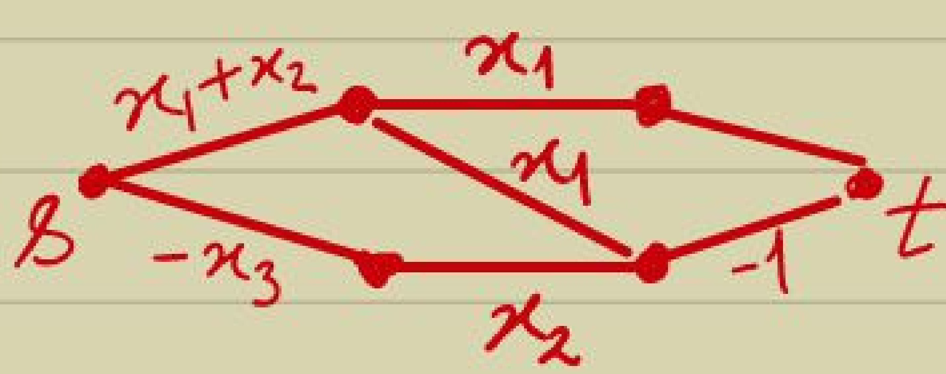
The polynomial computed by the ABP is $f = \sum_{\text{path } \gamma : s \rightsquigarrow t} wt(\gamma)$,

where $wt(\gamma)$ is the product of the edge weights in γ.

The width of the ABP is the max number of vertices in any layer.

The depth is the length of the max path from $s \rightsquigarrow t$.

- Eg.



computes $f = x_2 x_3$

in width $= 2$ & depth $= 3$.

- In this example, note that we can also represent f by using the adjacency matrices of the level transitions.

E.g. $[x_1+x_2, -x_3]\begin{bmatrix} x_1 & x_1 \\ 0 & x_2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = x_2x_3$.

Also $= [1, -1]\begin{bmatrix} x_1+x_2 & 0 \\ 0 & x_3 \end{bmatrix}\begin{bmatrix} x_1 & x_1 \\ 0 & x_2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Defn: Iterated matrix multiplication polynomial $IMM_{n,d}$ is the $(1,1)$-th entry of the product

$$X^{(1)} \cdot X^{(2)} \cdots\cdots X^{(d)},$$

where $X^{(i)}$ are $n \times n$ symbolic matrices (i.e. with each entry being a constant or a variable).

Theorem: If f has a width-w, depth-d ABP, then it has an IMM of size $O((wn)^2 \cdot d)$.

If f has an $IMM_{w,d}$ then it has a width-w, depth-d ABP.

Pf: (easy exercise)

▷ Any polynomial $f \in \mathbb{F}[x_1, \dots, x_n]$ of sparsity s & degree d has an ABP of size $O(sd)$.

Pf: Build the ABP for each monomial. $\square$

Defn: Symbolic determinant $D(\bar{x})$ is a polynomial that equals the determinant of an $m \times m$ matrix with entries as $\mathbb{F} \cup \bar{x}$. (m is the size)

- Our next big connection is the one between ABP & Symbolic determinant.

  For that, we need a graph interpretation of determinant.

- Any permutation $\sigma \in Sym(n)$ can be decomposed into cycles. Eg. $[5] \to (1,3,2,5,4)$ has the cycle decomposition: $(2\ 3)(4\ 5)$ with $\text{sign} = (-1)^{\#\text{even-cycles}} = (-1)^2 = 1$.

- In a graph G, a cycle cover is a

partition of $V(G)$ into cycles (simple disjoint).

<u>Theorem</u>: Let G be the graph on $V(G) = [n]$ with adjacency matrix $X = (x_{ij})_{n \times n}$. Then,

$$\det(X) = \sum_{C \in \mathrm{CycleCover}(G)} \mathrm{sgn}(C) \cdot \mathrm{wt}(C),$$

where $\mathrm{sgn}(C) := (-1)^{\#\text{even-cycles in } C}$ &

$$\mathrm{wt}(C) := \prod_{\text{edge } e \in C} \mathrm{wt}(e).$$

<u>Proof</u>: • We have $\det(X) = \sum_{\sigma \in \mathrm{Sym}(n)} \mathrm{sgn}(\sigma) \cdot \prod_{i \in [n]} x_{i,\sigma(i)}$.

• The summand corresponding to σ can be seen to be equal to $\mathrm{sgn}(C) \cdot \mathrm{wt}(C)$, where C is the cycle cover of G specified by the cycle decomposition of σ.

• Finally, note that <u>every</u> cycle cover C of G uniquely specifies some $\sigma \in \mathrm{Sym}(n)$. $\square$

Corollary: $per_n(X) = \sum_{C \in CycleCover(G)} wt(C)$.

- We are ready to reduce ABP to det.

Lemma: If f has a width-w, depth-d ABP, then it has a $O(wdn)$-size determinant.

Proof:

- We can first make the ABP edge weights symbolic, i.e. in $\mathbb{F} \cup \overline{x}$.

Ensure length d' paths $\rightarrow$ This makes the depth $O(d) =: d'$.

- Let G be the directed graph underlying this ABP. Modify it to a graph G':
  - Add a $wt=1$ edge from t to s.
  - On all other vertices add a $wt=1$ self-loop.
- Observe that any $C \in cycleCover(G')$ uniquely specifies a path from $s \rightsquigarrow t$, & the $sgn(C)$ is the same for all C (say 1).