Explicit expander constructions

- We will construct an explicit expander family.

- The main idea is to use three kinds of graph products.

  Tradeoff between degree & the spectral gap.

- For the operations we need a new representation of graphs:

Definition: If G is an n-vertex d-degree connected graph then we label each neighbour of each vertex using $[d]$.

Define a rotation map

$$\hat{G} : [n] \times [d] \longrightarrow [n] \times [d]$$

$$(v, i) \mapsto (u, j)$$, where u is the i-th vertex of v & v the j-th vertex of u.

▷ $\hat{G}$ is a permutation on $[n] \times [d]$.

Pf: It is 1-1 & onto. $\square$

Matrix Product (or Path product)

Definition: For two n-vertex graphs G, G', with degrees d, d', the graph GG' is the one with the normalized adjacency matrix AA'. (A, A' are the normalized adjacency matrices of G, G'.)

Proposition: 1) AA' is stochastic.
2) GG' may have repeated edges, the degree is dd', & #vertices $= n$.

▷ $\lambda(GG') \leq \lambda(G) \cdot \lambda(G')$.

Pf:

- $$\lambda(GG') = \max_{u \in \bar{1}^\perp} \frac{\|AA'u\|}{\|u\|} = \max_{u \in \bar{1}^\perp} \frac{\|AA'u\|}{\|A'u\|} \cdot \frac{\|A'u\|}{\|u\|}$$

$A'u \in \bar{1}^\perp$ →
$$\leq \lambda(G) \cdot \lambda(G').$$ $\square$

Theorem: If G, G' are (n,d,λ), (n',d',λ')-expanders then GG' is $(n, dd', \lambda\lambda')$-expander.

Pf: Clear. $\square$

$\Rightarrow$ Thus, matrix product improves the spectral gap at the cost of the degree.

Tensor product

Definition: The graph $\underset{n\text{-vertex}}{G} \otimes \underset{n'\text{-vertex}}{G'}$ is the one with the normalized adjacency matrix $A \otimes A'$,

where $$A\otimes A' := \begin{pmatrix} A_{1,1}A' & \cdots & A_{1,n}A' \\ \vdots & \vdots & \vdots \\ A_{n,1}A' & \cdots & A_{n,n}A' \end{pmatrix}_{nn' \times nn'}.$$

Proposition: 1) $A\otimes A'$ is symmetrix stochastic.

2) $G\otimes G'$ is nn'-vertex, dd'-degree.

▷ $\lambda(G \otimes G') = \max(\lambda(G), \lambda(G'))$.

Pf:

- Let $1 = \lambda_1 \geq |\lambda_2| \geq \dots \geq |\lambda_n|$ &
  $1 = \lambda'_1 \geq |\lambda'_2| \geq \dots \geq |\lambda'_n|$ be the eigenvalues of A & A' resp.
- Then the eigenvalues of $A \otimes A'$ are:
  $\{ \lambda_i \lambda'_j \mid i \in [n], j \in [n'] \}$.
  [$\because (A \otimes A') \cdot (v \otimes v') = Av \otimes A'v'$.]
- Thus, the largest ones, apart from 1, are
  $\{ \lambda_i \mid i \in [n] \} \cup \{ \lambda'_i \mid i \in [n'] \}$.
  $\Rightarrow \lambda(G \otimes G') = \max\{\lambda(G), \lambda(G')\}$. □

<u>Theorem</u>: If G, G' are $(n, d, \lambda), (n', d', \lambda')$-expanders, then $G \otimes G'$ is $(nn', dd', \max(\lambda, \lambda'))$- " .

$\Rightarrow$ Tensor product increase the #vertices while preserving the spectral gap.

The Replacement Product

- This product is easier to see as a walk rather than a matrix operation.

- Suppose we have an expander G with a high degree-D. In order to do a walk using fewer random bits, we can use another expander G' with vertex-D & degree-d ($\ll D$).

- This motivates the product $G \circledR G'$:

Definition: Let G, G' be graphs with vertices n, D, degrees D, d & normalized adjacency matrices A, A'.

$H := G \circledR G'$ is a nD-vertex graph s.t.

i) $\forall u \in V(G)$, H has a copy of G', say H_u; called a cloud. I.e. $\forall i \in V(G')$, $(u,i) \in V(H)$, & is called the i-th vertex in the u-th cloud.

ii) If $\hat{G}(u,i) = (v,j)$ then $((u,i),(v,j)) \in E(H)$.
Also, if $(i,j) \in E(G')$ then $\forall u \in V(G)$, $((u,i),(u,j)) \in E(H)$.

$\triangleright$ $H = G \circledR G'$ is nD-vertex, $(d+1)$-degree.

Pf:

- Any vertex in H corresponds to a vertex (u,i) in H_u, $u \in V(G)$.
- $\because$ $H_u \cong G'$, (u,i) has d neighbours in the cloud.
- Further property (ii) of the definition adds one edge outside the cloud.

$\Rightarrow$ H has degree $(d+1)$. $\square$

Zig-Zag Product

(has more memory)
(like self-loops)

- We consider length-3 paths that zig-zag the clouds.

Zig-Zag product

Definition: On the vertex set $V(G \circledR G')$ define $H := G \text{ⓩ} G'$ s.t. $((u,i),(v,j))$ is an edge if $\exists\, \ell, k$ with $((u,i),(u,\ell))$, $((u,\ell),(v,k))$, $((v,k),(v,j)) \in E(G \circledR G')$.

▷ $G \textcircled{z} G'$ is nD-vertex, d^2-degree.

▷ Its normalized adjacency matrix is $A \textcircled{z} A' := B\hat{A}B$, where

$$\hat{A}[(u,i),(v,j)] = \begin{cases} 1 & \text{, if } \hat{G}(u,i)=(v,j), \\ 0 & \text{, else.} \end{cases}$$

& $$B[(u,i),(v,j)] = \begin{cases} A'[i,j] & \text{, if } u=v, \\ 0 & \text{, else.} \end{cases}$$

Pf:

- $A \textcircled{z} A'$ encodes the definition of $G \textcircled{z} G'$.
- Also, $(A \textcircled{z} A') \cdot \begin{pmatrix}1\\ \vdots \\ 1\end{pmatrix} = B\hat{A}B \cdot \begin{pmatrix}1\\ \vdots \\ 1\end{pmatrix} = B\hat{A} \cdot \begin{pmatrix}1\\ \vdots \\ 1\end{pmatrix}$

$= B \cdot \begin{pmatrix}1\\ \vdots \\ 1\end{pmatrix}$ [$\because \hat{A}$ merely permutes]

$= \begin{pmatrix}1\\ \vdots \\ 1\end{pmatrix}$

$\Rightarrow$ $A \textcircled{z} A'$ is stochastic, symmetric. □

Theorem (Reingold, Vadhan, Wigderson, 2002): If $\lambda(G)=a$ & $\lambda(G')=b$ then $\lambda(G \textcircled{z} G') \le a+2b+b^2$.

<u>Proof</u>: • Let $M := A \textcircled{z} A'$.

• Recall that $M = B\hat{A}B$, where $\hat{A}$ is a

B is stochastic → permutation & $B = I_n \otimes A'$.

• Write $B = I_n \otimes \frac{1}{D}J + I_n \otimes \underbrace{\left(A' - \frac{1}{D}J\right)}_{E}$,

where J is the all-one matrix.

$\bar{J}$ is stochastic → • Define $\bar{J} := I_n \otimes \frac{1}{D}J$ & $\bar{E} := I_n \otimes E$.

$$\Rightarrow M = B\hat{A}B = (\bar{J}+\bar{E})\cdot\hat{A}\cdot(\bar{J}+\bar{E})$$

$$= \bar{J}\hat{A}\bar{J} + \bar{J}\hat{A}\bar{E} + \bar{E}\hat{A}\bar{J} + \bar{E}\hat{A}\bar{E}.$$

• For each of these matrices we will upper bound the matrix norm $\|A\| := \max\limits_{\substack{x \in 1^{\perp} \\ \|x\|=1}} \|Ax\|$.

$\lambda(A) =$

• Let $\underline{\mathcal{L}(A)} := \max\limits_{\|x\|=1} \|Ax\|$. equals largest eigen value

• Clearly, $\|E\| \leq \mathcal{L}(E) = \mathcal{L}\left(A' - \frac{J}{D}\right)$

• Let $\lambda_1, \dots, \lambda_D$ be the eigenvalues of A' s.t. $1 = |\lambda_1| \geq |\lambda_2| \geq \dots \geq |\lambda_D|$. We can write $A' = \sum_{i=1}^{D} \lambda_i v_i v_i^T$ using the orthonormal eigenvectors of A'.

$$\Rightarrow \|E\| \leq \mathcal{L}\left(A' - \frac{J}{D}\right) = \mathcal{L}\left(\sum_{i=2}^{D} \lambda_i v_i v_i^T\right) \leq |\lambda_2| = \|A'\| = b.$$

• Going back, we have by the triangle inequality:

$\|M\| \leq \|\bar{J}\hat{A}\bar{J}\| + \|\bar{J}\hat{A}\bar{E}\| + \|\bar{E}\hat{A}\bar{J}\| + \|\bar{E}\hat{A}\bar{E}\|$

$\leq \|\bar{J}\hat{A}\bar{J}\| + \|\bar{J}\| \cdot \|\hat{A}\| \cdot \|\bar{E}\| + \ldots\ldots$

[$\because$ $\hat{A}, \bar{J}, \bar{E}$ map $\bar{1}^{\perp}$ to $\bar{1}^{\perp}$.]

$\leq \|\bar{J}\hat{A}\bar{J}\| + 1 \cdot 1 \cdot b + b \cdot 1 \cdot 1 + b \cdot 1 \cdot b$

• Now we estimate $\|\bar{J}\hat{A}\bar{J}\|$.

▷ $\bar{J}\hat{A}\bar{J} = A \otimes D^{-1} \cdot J$.

Pf: • $(\bar{J}\hat{A}\bar{J})_{(u,i),(v,j)} = \bar{J}_{(u,i),-} \cdot \hat{A} \cdot \bar{J}_{-,(v,j)}$

$= \frac{1}{D} \cdot A_{u,v} = (A \otimes D^{-1} \cdot J)_{(u,i),(v,j)}$. □

• Now note that the eigenvalues of J are $\{1, 0, \ldots, 0\}$.

This is because $J \cdot \bar{1} = \bar{1}$ & $J \cdot \bar{x} = 0$, $\forall \bar{x} \in \bar{1}^{\perp}$.

$\Rightarrow \|A \otimes D^{-1} \cdot J\| = \|A\| = a$.

$\Rightarrow \|\bar{J}\hat{A}\bar{J}\| = a$.

$$\Rightarrow \|M\| \leq a + 2b + b^2 . \qquad \square$$

<u>Theorem</u>: If G, G' are (n, D, λ_1), (D, d, λ_2)-expanders then $G ⓩ G'$ is $(nD, d^2, \lambda_1 + 2\lambda_2 + \lambda_2^2)$- " .

- Thus, it reduces the degree dramatically when $D \gg d$, without increasing the λ too much.

<u>The Construction</u>

- Using the three products we now give a strongly <u>explicit</u> construction of expanders,

  i.e. given (u, i) the i-th neighbour of u can be computed in $poly(\lg |V|)$ time.

<u>Theorem</u>: $\exists$ strongly explicit (d^2, λ)-expander family for ooly-many constants $d \in \mathbb{N}$ & $\lambda \in (0,1)$.

Pf: • We will recursively construct $\{G_k\}_{k \geq 1}$ s.t. G_k has $2^{O(k)}$ vertices.

• Let H be a $(d^8, d, 0.04)$-expander.

It can be found by the known constructions, or randomly, for a constant d.

• Also, let $G_1 := H^2$ be $(d^8, d^2, 0.04^2)$-expander.

• Initialize $G_2 = G_1$.

• For odd $k > 2$, $G_{k+1} = G_k := \left(G_{\frac{k-1}{2}}^{\otimes 2}\right)^2 ⓩ H$.

Claim: $\forall$ odd k, G_k is $(d^{8k}, d^2, 0.1)$-expander.

Pf:

• True for $k = 1$.

• By induction, #vertices $(G_k) =$

$$d^{8 \cdot \frac{k-1}{2} \times 2} \cdot d^8 = d^{8k}.$$

• Similarly, $\deg \left(G_{\frac{k-1}{2}}^{\otimes 2}\right)^2 = d^{2 \cdot 2 \cdot 2} = d^8 = |V(H)|$

$\Rightarrow$ the zig-zag product is defined & reduces the degree to d^2.

• Finally, $\lambda(G_k) \leq 0.1^2 + 2 \cdot (0.04) + (0.04)^2$

$$< 0.1 \quad .$$

$\square$

- The algorithm for finding the neighbours of $v \in V(G_k)$ is <u>recursive</u>:

Say, listing a row in $G_{\frac{k-1}{2}}$ takes time $T(\frac{k-1}{2})$.

$\Rightarrow$ We can list a row in $G_{\frac{k-1}{2}}^{\otimes 2}$ in time $2 \cdot T(\frac{k-1}{2})$. (It's d^4-sparse.)

$\Rightarrow$ Listing all the needed d^4-rows in the other $G_{\frac{k-1}{2}}^{\otimes 2}$ takes $d^4 \cdot 2 \cdot T(\frac{k-1}{2})$ time.

$\Rightarrow T(k) = O\left(d^4 \cdot T(\frac{k-1}{2})\right)$

$\Rightarrow T(k) = d^{O(\lg k)} = k^{O(\lg d)}$

$= \text{poly}(k) = \text{poly-log}(\# V(G_k))$.

const. d $\square$

— Now, before we derandomize "Upath $\in$ RL" we will need another estimate for $\lambda(G \otimes G')$.

Because $\lambda(G)$ is, in general, large & then the previous bound is useless.

Theorem (Reingold, Trevisan, Vadhan 2005): If $\lambda(G)=1-\varepsilon$ & $\lambda(G')=1-\delta$ then $\lambda(G \textcircled{z} G') \leq 1-\varepsilon\delta^2$.

Proof:

- As before, $M := A \textcircled{z} A' = B\hat{A}B$, where $B = I_n \otimes A'$ & $\hat{A}$ is the rotation map of G.
- We can express A' as $D^{-1}\delta J + (1-\delta)C$, where J is the all-1 matrix & C is a matrix s.t. $\|C\| \leq 1$.

Again use $A' = \sum_{i=1}^{D} \lambda_i v_i v_i^T$

(Show that $\|\frac{1}{1-\delta}\cdot(A'-\frac{\delta}{D}J)\cdot x\| \leq \|x\|$.)

$\Rightarrow B = I_n \otimes (\delta D^{-1} J + (1-\delta)C) =: \delta\cdot\bar{J} + (1-\delta)\cdot\bar{C}$

$\Rightarrow M = (\delta\bar{J} + (1-\delta)\bar{C})\cdot\hat{A}\cdot(\delta\bar{J} + (1-\delta)\bar{C})$

$= \delta^2\cdot\bar{J}\hat{A}\bar{J} + (1-\delta^2)\cdot F$

where $F := \frac{\delta}{1+\delta}\cdot(\bar{J}\hat{A}\bar{C} + \bar{C}\hat{A}\bar{J}) + \frac{1-\delta}{1+\delta}\cdot\bar{C}\hat{A}\bar{C}$

with $\|F\| \leq 1$. ($\because \|C\| \leq 1$, $\hat{A}$ is a permutation)

$\Rightarrow M = \delta^2\cdot A \otimes D^{-1}J + (1-\delta^2)\cdot F$

$\Rightarrow \lambda(M) = \|M\| \leq \delta^2\cdot\|A \otimes D^{-1}J\| + (1-\delta^2)\cdot\|F\|$

$\leq \delta^2\cdot(1-\varepsilon) + (1-\delta^2)\cdot 1$

$= (1-\varepsilon\delta^2)$.

$\square$

<u>Theorem</u> (Reingold 2005): Upath $\in \mathbb{L}$.

<u>Proof</u>:

- Let G be the given undirected n-vertex graph (s be the start vertex).
- <u>Idea</u>: To apply the graph products on G to get $\tilde{G}$ s.t. the connected component of s in $\tilde{G}$ is an expander (constant λ, d).

  The shortest paths in $\tilde{G}$ are then only $O(\lg n)$ in length.

- Let d be a large enough constant s.t. a $(d^{16}, d, 1/2)$-expander H exists.
- Wlog assume G to be $D := d^{16}$-regular.
- Now we transform G as follows:

$$G_0 := G \,,\quad G_{i+1} := (G_i \circledcirc H)^8 \text{ for } i \geq 0.$$

▷ G_k is an (nd^{16k}, d^{16})-graph.

<u>Pf</u>: • by induction, $\#V(G_k) = nd^{16(k-1)} \cdot d^{16} = nd^{16k}$, & $\deg(G_k) = (d^2)^8 = d^{16}$. □

- We have seen before that for any graph,
$$\lambda(G) \leq 1 - \frac{1}{8 \cdot D \cdot n^3} = 1 - \frac{1}{8 \cdot d^{16} \cdot n^3}.$$

- If $\lambda(G_{k-1}) =: 1-\varepsilon$ then we have
$$\lambda(G_k) \leq \left(1 - \varepsilon \cdot \frac{1}{4}\right)^8$$

$$\Rightarrow 1 - \lambda(G_k) \geq 1 - \left(1 - \frac{\varepsilon}{4}\right)^8 = \binom{8}{1} \cdot \frac{\varepsilon}{4} - \binom{8}{2} \cdot \left(\frac{\varepsilon}{4}\right)^2 + \ldots\ldots$$

- Note that for $\varepsilon < 1/2$, the above expression means that the spectral gap increases by a multiple of $9/8$.

$\Rightarrow$ For $\ell = O(\lg n)$, $1 - \lambda(G_\ell) \geq 1/2$.

- Thus, G_ℓ has $O(\lg n)$-length shortest paths & constant degree d^{16}.

- All that remains is to show that the walk in G_ℓ is possible in $O(\lg n)$-space.

(Use Reingold's recursive data structure.) $\square$