Pf: • We will recursively construct $\{G_k\}_{k \geq 1}$ s.t. G_k has $2^{O(k)}$ vertices.

• Let H be a $(d^8, d, 0.04)$-expander. It can be found by the known constructions, or randomly, for a constant d.

• Also, let $G_1 := H^2$ be $(d^8, d^2, 0.04^2)$-expander.

• Initialize $G_2 = G_1$.

• For odd $k > 2$, $G_{k+1} = G_k := \left(G_{\frac{k-1}{2}}^{\otimes 2}\right)^2 ⓩ H$.

Claim: $\forall$ odd k, G_k is $(d^{8k}, d^2, 0.1)$-expander.

Pf:

• True for $k=1$.

• By induction, #vertices$(G_k) = d^{8 \cdot \frac{k-1}{2} \times 2} \cdot d^8 = d^{8k}$.

• Similarly, $\deg\left(G_{\frac{k-1}{2}}^{\otimes 2}\right)^2 = d^{2 \cdot 2 \cdot 2} = d^8 = |V(H)|$

$\Rightarrow$ the zig-zag product is defined & reduces the degree to d^2.

• Finally, $\lambda(G_k) \leq 0.1^2 + 2 \cdot (0.04) + (0.04)^2 < 0.1$. $\square$

- The algorithm for finding the neighbours of $v \in V(G_k)$ is <u>recursive</u>:

Say, listing a row in $G_{\frac{k-1}{2}}$ takes time $T(\frac{k-1}{2})$.

$\Rightarrow$ We can list a row in $G_{\frac{k-1}{2}}^{\otimes 2}$ in time $2 \cdot T(\frac{k-1}{2})$. (It's d^4-sparse.)

$\Rightarrow$ Listing all the needed d^4-rows in the other $G_{\frac{k-1}{2}}^{\otimes 2}$ takes $d^4 \cdot 2 \cdot T(\frac{k-1}{2})$ time.

$\Rightarrow$ $T(k) = O\left(d^4 \cdot T(\frac{k-1}{2})\right)$

$\Rightarrow$ $T(k) = d^{O(\lg k)} = k^{O(\lg d)}$

$= \text{poly-log}(\# V(G_k))$.

const. d $\square$

— Now, before we derandomize "Upath $\in$ RL" we will need another estimate for $\lambda(G \circledz G')$.

Because $\lambda(G)$ is, in general, large & then the previous bound is useless.

<u>Theorem</u> (Reingold, Trevisan, Vadhan 2005): If $\lambda(G)=1-\varepsilon$ & $\lambda(G')=1-\delta$ then $\lambda(G \textcircled{z} G') \le 1-\varepsilon\delta^2$.

<u>Proof</u>:

- As before, $M := A \textcircled{z} A' = B\hat{A}B$, where $B = I_n \otimes A'$ & $\hat{A}$ is the rotation map of G.
- We can express A' as $D^{-1}\delta J + (1-\delta)C$, where J is the all-1 matrix & C is a matrix s.t. $\|C\| \le 1$.

(Show that $\|\frac{1}{1-\delta}\cdot(A' - \frac{\delta}{D}J)\cdot x\| \le \|x\|$.)

$$\Rightarrow B = I_n \otimes (\delta D^{-1} J + (1-\delta)C) =: \delta\cdot\bar{J} + (1-\delta)\cdot\bar{C}$$

$$\Rightarrow M = (\delta\bar{J} + (1-\delta)\bar{C})\cdot\hat{A}\cdot(\delta\bar{J} + (1-\delta)\bar{C})$$

$$= \delta^2\cdot\bar{J}\hat{A}\bar{J} + (1-\delta^2)\cdot F$$

where $F := \frac{\delta}{1+\delta}\cdot(\bar{J}\hat{A}\bar{C} + \bar{C}\hat{A}\bar{J}) + \frac{1-\delta}{1+\delta}\cdot\bar{C}\hat{A}\bar{C}$

with $\|F\| \le 1$. ($\because \|C\| \le 1$, $\hat{A}$ is a permutation)

$$\Rightarrow M = \delta^2\cdot A \otimes D^{-1}J + (1-\delta^2)\cdot F$$

$$\Rightarrow \lambda(M) = \|M\| \le \delta^2\cdot\|A \otimes D^{-1}J\| + (1-\delta^2)\cdot\|F\|$$

$$\le \delta^2\cdot(1-\varepsilon) + (1-\delta^2)\cdot 1$$

$$= (1-\varepsilon\delta^2).$$

$\square$

Theorem (Reingold 2005): Upath $\in \mathbb{L}$.

Proof:

- Let G be the given undirected n-vertex graph (s be the start vertex).
- **Idea:** To apply the graph products on G to get $\tilde{G}$ s.t. the connected component of s in $\tilde{G}$ is an expander (constant λ, d).

  The shortest paths in $\tilde{G}$ are then only $O(\lg n)$ in length.

- Let d be a large enough constant s.t. a $(d^{16}, d, 1/2)$-expander H exists.
- Wlog assume G to be $D := d^{16}$-regular.
- Now we transform G as follows:

$$G_0 := G \,,\quad G_{i+1} := (G_i \circledz H)^8 \text{ for } i \geq 0.$$

▷ G_k is an (nd^{16k}, d^{16})-graph.

Pf: • by induction, $\#V(G_k) = nd^{16(k-1)} \cdot d^{16} = nd^{16k}$,

& $\deg(G_k) = (d^2)^8 = d^{16}$. □

- We have seen before that for any graph,
$$\lambda(G) \leq 1 - \frac{1}{8 \cdot D \cdot n^3} = 1 - \frac{1}{8 \cdot d^{16} \cdot n^3}.$$

- If $\lambda(G_{k-1}) =: 1-\varepsilon$ then we have
$$\lambda(G_k) \leq \left(1 - \varepsilon \cdot \frac{1}{4}\right)^8$$

$$\Rightarrow 1 - \lambda(G_k) \geqslant 1 - \left(1 - \frac{\varepsilon}{4}\right)^8 = \binom{8}{1} \cdot \frac{\varepsilon}{4} - \binom{8}{2} \cdot \left(\frac{\varepsilon}{4}\right)^2 + \ldots\ldots$$

- Note that for $\varepsilon < 1/2$, the above expression means that the spectral gap increases by a multiple of $9/8$.

$\Rightarrow$ For $l = O(\lg n)$, $1 - \lambda(G_l) \geqslant 1/2$.

- Thus, G_l has $O(\lg n)$-length shortest paths & constant degree d^{16}.

- All that remains is to show that the walk in G_l is possible in $O(\lg n)$-space.

(Use Reingold's recursive data structure.) $\square$