The Replacement Product

- This product is easier to see as a walk rather than a matrix operation.

- Suppose we have an expander G with a high degree-D. In order to do a walk using fewer random bits, we can use another expander G' with vertex-D & degree-d $(\ll D)$.

- This motivates the product $G \circledR G'$:

Definition: Let G, G' be graphs with vertices n, D, degrees D, d & normalized adjacency matrices A, A'.

$H := G \circledR G'$ is a nD-vertex graph s.t.

i) $\forall u \in V(G)$, H has a copy of G', say H_u; called a cloud. I.e. $\forall i \in V(G')$, $(u,i) \in V(H)$, & is called the i-th vertex in the u-th cloud.

ii) If $\hat{G}(u,i) = (v,j)$ then $((u,i),(v,j)) \in E(H)$.
Also, if $(i,j) \in E(G')$ then $\forall u \in V(G)$, $((u,i),(u,j)) \in E(H)$.

$\triangleright$ $H = G \circledR G'$ is nD-vertex, $(d+1)$-degree.

Pf:

- Any vertex in H corresponds to a vertex (u,i) in H_u, $u \in V(G)$.
- $\because$ $H_u \cong G'$, (u,i) has d neighbours in the cloud.
- Further property (ii) of the definition adds one edge outside the cloud.

$\Rightarrow$ H has degree $(d+1)$. $\square$

Zig-Zag Product

- We consider length-3 paths that zig-zag the clouds.

Zig-Zag product

Definition: On the vertex set $V(G \circledR G')$ define $H := G \circledz G'$ s.t. $((u,i),(v,j))$ is an edge if $\exists \, l,k$ with $((u,i),(u,l))$, $((u,l),(v,k))$, $((v,k),(v,j)) \in E(G \circledR G')$.

▷ $G \textcircled{z} G'$ is nD-vertex, d^2-degree.

▷ Its normalized adjacency matrix is $A \textcircled{z} A' := B\hat{A}B$, where

$$\hat{A}[(u,i),(v,j)] = \begin{cases} 1 & , \text{ if } \hat{G}(u,i) = (v,j), \\ 0 & , \text{ else.} \end{cases}$$

& $$B[(u,i),(v,j)] = \begin{cases} A'[i,j] & , \text{ if } u = v, \\ 0 & , \text{ else.} \end{cases}$$

Pf:

- $A \textcircled{z} A'$ encodes the definition of $G \textcircled{z} G'$.
- Also, $(A \textcircled{z} A') \cdot \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = B\hat{A}B \cdot \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = B\hat{A} \cdot \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$

$$= B \cdot \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \quad [\because \hat{A} \text{ merely permutes}]$$
$$= \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

$\Rightarrow$ $A \textcircled{z} A'$ is stochastic, symmetric. $\square$

Theorem (Reingold, Vadhan, Wigderson, 2002): If $\lambda(G) = a$ & $\lambda(G') = b$ then $\lambda(G \textcircled{z} G') \leq a + 2b + b^2$.

Proof: • Let $M := G \textcircled{z} G'$.

• Recall that $M = B\hat{A}B$, where $\hat{A}$ is a permutation & $B = I_n \otimes A'$.

B is stochastic

• Write $B = I_n \otimes \frac{1}{D} J + I_n \otimes \underbrace{\left(A' - \frac{1}{D} J\right)}_{E}$, where J is the all-one matrix.

$\bar{J}$ is stochastic

• Define $\bar{J} := I_n \otimes \frac{1}{D} J$ & $\bar{E} := I_n \otimes E$.

$$\Rightarrow M = B\hat{A}B = (\bar{J}+\bar{E}) \cdot \hat{A} \cdot (\bar{J}+\bar{E})$$
$$= \bar{J}\hat{A}\bar{J} + \bar{J}\hat{A}\bar{E} + \bar{E}\hat{A}\bar{J} + \bar{E}\hat{A}\bar{E}.$$

• For each of these matrices we will upper bound the matrix norm $\lambda(A) = \|A\| := \max\limits_{\substack{x \in \bar{1}^{\perp} \\ \|x\|=1}} \|Ax\|$.

• First, consider $\bar{E}$.

• Let $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)^T \in (\mathbb{R}^D)^n$ be a vector.

$$\bullet\ \|\bar{E}\bar{x}\|^2 = \|(I_n \otimes E) \cdot (\bar{x}_1, \ldots, \bar{x}_n)^T\|^2$$
$$= \sum_{u \in [n]} \|E\bar{x}_u\|^2 \underset{\uparrow \text{ defn. of } \|E\|}{\leq} \sum_u \|E\|^2 \cdot \|\bar{x}_u\|^2$$
$$= \|E\|^2 \cdot \|\bar{x}\|^2$$
$$\Rightarrow \|\bar{E}\| \leq \|E\| = \|A' - \tfrac{1}{D} J\| = \|A'\| = b.$$

• Going back, we have by the triangle inequality:

$$\|M\| \leq \|\bar{J}\hat{A}\bar{J}\| + \|\bar{J}\hat{A}\bar{E}\| + \|\bar{E}\hat{A}\bar{J}\| + \|\bar{E}\hat{A}\bar{E}\|$$

$$\leq \|\bar{J}\hat{A}\bar{J}\| + \|\bar{J}\|\cdot\|\hat{A}\|\cdot\|\bar{E}\| + \ldots\ldots$$

[$\because$ $\hat{A}, \bar{J}, \bar{E}$ map $\bar{1}^{\perp}$ to $\bar{1}^{\perp}$.]

$$\leq \|\bar{J}\hat{A}\bar{J}\| + 1\cdot 1\cdot b + b\cdot 1\cdot 1 + b\cdot 1\cdot b$$

• Now we estimate $\|\bar{J}\hat{A}\bar{J}\|$.

▷ $\bar{J}\hat{A}\bar{J} = A \otimes D^{-1}\cdot J$.

Pf: • $(\bar{J}\hat{A}\bar{J})_{(u,i),(v,j)} = \bar{J}_{(u,i),-} \cdot \hat{A} \cdot \bar{J}_{-,(v,j)}$

$$= \frac{1}{D}\cdot A_{u,v} = (A \otimes D^{-1} J)_{(u,i),(v,j)}. \qquad \square$$

• Now note that the eigenvalues of J are 1 & 0.

This is because $J\cdot\bar{1} = \bar{1}$ & $J\cdot\bar{x} = 0$, $\forall \bar{x} \in \bar{1}^{\perp}$.

$\Rightarrow$ $\|A \otimes D^{-1}\cdot J\| = \|A\| = a$.

$\Rightarrow$ $\|\bar{J}\hat{A}\bar{J}\| = a$.

$$\Rightarrow \|M\| \leq a + 2b + b^2. \qquad \square$$

<u>Theorem</u>: If G, G' are (n, D, λ_1), (D, d, λ_2)-expanders then $G ⓩ G'$ is $(nD, d^2, \lambda_1 + 2\lambda_2 + \lambda_2^2)$- " .

- Thus, it reduces the degree dramatically when $D \gg d$, without increasing the λ too much.

<u>The Construction</u>

- Using the three products we now give a strongly <u>explicit</u> construction of expanders,

  i.e. given (u, i) the i-th neighbour of u can be computed in poly$(\lg|V|)$ time.

<u>Theorem</u>: $\exists$ <u>strongly explicit</u> (d, λ)-expander family for only-many constants $d \in \mathbb{N}$ & $\lambda \in (0,1)$.