Explicit expander constructions

- We will construct an explicit expander family.

- The main idea is to use three kinds of graph products.
  Tradeoff between degree & the spectral gap.

- For the operations we need a new representation of graphs:

Definition: If G is an n-vertex d-degree connected graph then we label each neighbour of each vertex using $[d]$.

Define a rotation map

$$\hat{G} : [n] \times [d] \longrightarrow [n] \times [d]$$

$$(v, i) \mapsto (u, j)$$, where u is the i-th vertex of v & v the j-th vertex of u.

▷ $\hat{G}$ is a permutation on $[n] \times [d]$.

Pf: It is 1-1 & onto. $\square$

Matrix Product (or Path product)

Definition: For two n-vertex graphs G, G', with degrees d, d', the graph GG' is the one with the normalized adjacency matrix AA'. (A, A' are the normalized adjacency matrices of G, G'.)

Proposition: 1) AA' is stochastic.

2) GG' may have repeated edges, the degree is dd', & #vertices $= n$.

▷ $\lambda(GG') \leq \lambda(G) \cdot \lambda(G')$.

Pf:

- $$\lambda(GG') = \max_{u \in \bar{1}^\perp} \frac{\|AA'u\|}{\|u\|} = \max_{u \in \bar{1}^\perp} \frac{\|AA'u\|}{\|A'u\|} \cdot \frac{\|A'u\|}{\|u\|}$$

$A'u \in \bar{1}^\perp \rightarrow$ $\leq \lambda(G) \cdot \lambda(G')$. $\square$

Theorem: If G, G' are (n,d,λ), (n',d',λ')-expanders then GG' is $(n, dd', \lambda\lambda')$-expander.

Pf: Clear. $\square$

$\Rightarrow$ Thus, matrix product improves the spectral gap at the cost of the degree.

Tensor product

Definition: The graph $\underset{\text{n-vertex}}{G} \otimes \underset{\text{n'-vertex}}{G'}$ is the one with the normalized adjacency matrix $A \otimes A'$,

where $A \otimes A' := \begin{pmatrix} A_{1,1} A' & \cdots & A_{1,n} A' \\ \vdots & \vdots & \vdots \\ A_{n,1} A' & \cdots & A_{n,n} A' \end{pmatrix}_{nn' \times nn'}$.

Proposition: 1) $A \otimes A'$ is symmetrix stochastic.

2) $G \otimes G'$ is nn'-vertex, dd'-degree.

▷ $\lambda(G \otimes G') = \max(\lambda(G), \lambda(G'))$.

Pf.:

- Let $1 = \lambda_1 \geq |\lambda_2| \geq \dots \geq |\lambda_n|$ &

  $1 = \lambda'_1 \geq |\lambda'_2| \geq \dots \geq |\lambda'_n|$ be the eigenvalues of A & A' resp.
- Then the eigenvalues of $A \otimes A'$ are:

  $\{\lambda_i \lambda'_j \mid i \in [n], j \in [n']\}$.

  [$\because (A \otimes A') \cdot (v \otimes v') = Av \otimes A'v'$.]
- Thus, the largest ones, apart from 1, are

  $\{\lambda_i \mid i \in [n]\} \cup \{\lambda'_i \mid i \in [n']\}$.

  $\Rightarrow \lambda(G \otimes G') = \max\{\lambda(G), \lambda(G')\}$. □

<u>Theorem</u>: If G, G' are (n, d, λ), (n', d', λ')-expanders, then $G \otimes G'$ is $(nn', dd', \max(\lambda, \lambda'))$- " .

$\Rightarrow$ Tensor product increase the #vertices while preserving the spectral gap.