- We will now show the equivalence of the two definitions. (G is regular, has self-loops)

Theorem 1: G is an (n,d,λ)-expander $\Rightarrow$
" " " $(n,d,\frac{1-\lambda}{2})$-edge expander.

Theorem 2: G is an (n,d,ρ)-edge expander $\Rightarrow$
" " " $(n,d,1-\rho^2/2)$-expander.

▷ Cheeger's inequality: $\frac{1-\lambda(G)}{2} \leq \rho(G) \leq \sqrt{2(1-\lambda(G))}$.

Pf of Thm 1:

- The number of edges going out of S are estimated by considering $Z := \sum_{i,j\in[n]} A_{ij} \cdot (x_i - x_j)^2$.

- Define $\bar{x} \in \mathbb{R}^n$ as $x_i := \begin{cases} |\bar{S}| & \text{if } i \in S \\ -|S| & \text{if } i \notin S. \end{cases}$

$$\Rightarrow Z = \sum_{(i,j)\in S^2} + \sum_{(i,j)\in \bar{S}^2} + \sum_{(i,j)\in S\times\bar{S} \cup \bar{S}\times S}$$

$$= 0 + 0 + 2\cdot n^2 \cdot \sum_{(i,j)\in S\times\bar{S}} A_{ij} = 2n^2 \cdot \frac{\#E(S,\bar{S})}{d}.$$

- On the other hand,

$$Z = \sum_{i,j} A_{i,j} x_i^2 - 2 \cdot \sum_{i,j} A_{i,j} x_i x_j + \sum_{i,j} A_{ij} x_j^2$$

$$= \sum_i \left(\sum_j A_{ij}\right) \cdot x_i^2 - 2 \langle A\bar{x}, \bar{x} \rangle + \sum_j \left(\sum_i A_{ij}\right) x_j^2$$

$$= 2 \cdot \|\bar{x}\|^2 - 2 \cdot \langle A\bar{x}, \bar{x} \rangle$$

$$\geq 2 \cdot \|\bar{x}\|^2 - 2 \cdot \lambda \|x\|^2$$

$(\because \bar{x} \in \bar{1}^{\perp} \Rightarrow \|A\bar{x}\| \leq \lambda \|\bar{x}\|$

& Cauchy-Schwarz : $|\langle \bar{x}, \bar{y} \rangle| \leq \|\bar{x}\| \cdot \|\bar{y}\|$)

$$\Rightarrow \#E(S, \bar{S}) \cdot \frac{2n^2}{d} = Z \geq 2 \cdot \|x\|^2 \cdot (1-\lambda)$$

$$= 2 \cdot (1-\lambda) \cdot \left(|S| \cdot |\bar{S}|^2 + |\bar{S}| \cdot |S|^2\right)$$

$$\Rightarrow \#E(S, \bar{S}) \geq \frac{(1-\lambda) d}{n} \cdot |S| \cdot |\bar{S}|$$

$$\geq (1-\lambda) \cdot \frac{d}{n} \cdot |S| \cdot \frac{n}{2} = \left(\frac{1-\lambda}{2}\right) \cdot d \cdot |S| .$$

$\Rightarrow$ G is an $\left(n, d, \frac{1-\lambda}{2}\right)$-edge expander.

$\square$

Pf of Thm 2: Now we assume G to be an (n,d,ρ)-edge expander.

- The idea again is to estimate Z, but now using an eigenvector corresponding to λ_2 as $\bar{x}$.

Recall that $1=\lambda_1 \geq |\lambda_2| \geq \dots \geq |\lambda_n|$ are the eigenvalues of A.

- Let $\bar{u} \neq 0$ be an eigenvector s.t.

$$A\bar{u} = \lambda_2 \cdot \bar{u} \quad \& \quad \bar{u} \in \bar{1}^{\perp}.$$

- Since $\bar{u}$ has positive & negative coordinates, let us collect them in $\bar{v}$ & $\bar{w}$ respectively.

$$\Rightarrow \bar{u} = \bar{v} + \bar{w} \ ; \ \bar{v}, -\bar{w} \in (\mathbb{R}_{\geq 0})^n.$$

- Wlog $\bar{v}$ has $\leq \frac{n}{2}$ nonzero entries (else we take $-\bar{u}$).

Assuming $v_1 \geq v_2 \geq \dots \geq v_n \geq 0$

- Consider $Z := \sum_{i<j \in [n]} A_{ij} \cdot (v_i^2 - v_j^2)$.

- We will show that:

Claim 1: $Z \geqslant \rho \cdot \|\bar{v}\|^2$ (uses edge expansion)

Claim 2: $Z \leq \sqrt{2(1-\lambda_2)} \cdot \|\bar{v}\|^2$ (general graph property)

▷ Clearly, the above two claims prove Thm 2.

Pf of Claim 1:

- Sort the coordinates of $\bar{v}$: $v_1 \geqslant \dots \geqslant v_n \geqslant 0$ with $v_i = 0$ for $i > n/2$.
- $Z = \sum_{i < j \in [n]} A_{ij} (v_i^2 - v_j^2)$

$$= \sum_{i<j\in[n]} A_{ij} \sum_{i \leq k < j} (v_k^2 - v_{k+1}^2)$$

(flip the sums)

$$= (1/d) \cdot \sum_{k=1}^{n/2} \#E([k], [k+1 \dots n]) \cdot (v_k^2 - v_{k+1}^2)$$

(Using the fact that $v_i = 0$, $i > n/2$)

$$\geqslant (1/d) \cdot \sum_{1 \leq k \leq n/2} \rho d k \cdot (v_k^2 - v_{k+1}^2) = \rho \cdot \sum_{1 \leq k \leq \frac{n}{2}} (k v_k^2 - k v_{k+1}^2)$$

$$= \rho \cdot \sum_{1 \leq k \leq \lfloor n/2 \rfloor} (k v_k^2 - (k-1) v_k^2) = \rho \cdot \|v\|^2. \quad \square$$

Pf of Claim 2:

- Z & λ_2 are fundamentally related:

- $\langle A\bar{u}, \bar{v}\rangle = \langle \lambda_2 \bar{u}, \bar{v}\rangle = \langle \lambda_2 \bar{v} + \lambda_2 \bar{w}, \bar{v}\rangle = \lambda_2 \cdot \|\bar{v}\|^2.$
- Also, $\langle A\bar{u}, \bar{v}\rangle = \langle A\bar{v}, \bar{v}\rangle + \langle A\bar{w}, \bar{v}\rangle \leq \langle A\bar{v}, \bar{v}\rangle.$

$$\Rightarrow \quad \lambda_2 \leq \frac{\langle A\bar{v}, \bar{v}\rangle}{\|\bar{v}\|^2}$$

$$\Rightarrow 1-\lambda_2 \geqslant \frac{\|\bar{v}\|^2 - \langle A\bar{v}, \bar{v}\rangle}{\|\bar{v}\|^2}$$

$$= \frac{2\cdot \sum_i v_i^2 - \sum_{i,j} 2\cdot A_{ij}\cdot v_i \cdot v_j}{2\cdot \|\bar{v}\|^2}$$

$$= \frac{\sum_{i,j} A_{ij} v_i^2 + \sum_{i,j} A_{ij} v_j^2 - \sum_{i,j} 2A_{ij}\cdot v_i v_j}{2\cdot \|\bar{v}\|^2}$$

$$= 2\cdot \sum_{i<j\in[n]} A_{ij}\cdot (v_i - v_j)^2 \Big/ 2\cdot \|\bar{v}\|^2$$

$$= \frac{\left\{\sum_{i<j} A_{ij}\cdot (v_i - v_j)^2\right\}\cdot \left\{\sum_{i<j} A_{ij}\cdot (v_i + v_j)^2\right\}}{\|\bar{v}\|^2 \cdot \left\{ \quad '' \quad \right\}}$$

- Let us estimate the two expressions:

- Numerator — Cauchy-Schwarz inequality gives: $\geq \left(\sum_{i<j} A_{ij} \cdot (v_i^2 - v_j^2) \right)^2 = Z^2$.

- Denominator —

$$\tfrac{1}{2} \cdot \sum_{i,j \in [n]} A_{ij} \cdot (v_i + v_j)^2 = \frac{1}{2} \sum A_{ij} \cdot (v_i^2 + v_j^2) + \sum A_{ij} v_i v_j$$

$$= \|\bar{v}\|^2 + \sum A_{ij} v_i v_j$$

$$\leq \|\bar{v}\|^2 + \sum A_{ij} \left(\frac{v_i^2 + v_j^2}{2} \right) = 2 \cdot \|\bar{v}\|^2$$

$$\Rightarrow 1 - \lambda_2 \geq Z^2 / 2 \cdot \|\bar{v}\|^4$$

$$\Rightarrow Z \leq \sqrt{2(1-\lambda_2)} \cdot \|\bar{v}\|^2$$

□

- The polynomial $Z(G) = \sum_{i,j \in [n]} A_{ij} \cdot (x_i - x_j)^2$ is called <u>Laplacian quadratic form</u> of G. It carries useful information about <u>expansion</u> & <u>sparsest cut</u> in G.