Pf: • Let $u \in \mathbb{1}^{\perp}$ be a unit vector & $v := Au$.

• We will show $1 - \|v\|^2 \geq 1/4dn^3$.

$$\text{Thus, } \|v\|^2 \leq 1 - \frac{1}{4dn^3}$$

$$\Rightarrow \|v\| \leq 1 - 1/8dn^3 .$$

▷ $1 - \|v\|^2 = \sum_{i,j \in [n]} A_{ij} \cdot (u_i - v_j)^2$.

Pf:

$$\begin{aligned}
\bullet \sum A_{ij} \cdot (u_i - v_j)^2 &= \sum A_{ij} u_i^2 - 2 \sum A_{ij} u_i v_j + \sum A_{ij} v_j^2 \\
&= \sum_i \Big(\sum_j A_{ij}\Big) \cdot u_i^2 - 2 \cdot \langle Au, v \rangle + \sum_j \Big(\sum_i A_{ij}\Big) \cdot v_j^2 \\
&= \|u\|^2 - 2 \cdot \|v\|^2 + \|v\|^2 \\
&= 1 - \|v\|^2 .
\end{aligned}$$

□

• Thus, it suffices to show that $\exists\, i,j$ st.

$$A_{ij} \cdot (u_i - v_j)^2 \geq 1/4dn^3 .$$

• If $\exists\, i$, $(u_i - v_i)^2 \geq 1/4n^3$ then we are done (as $A_{ii} = 1/d$).

• So, assume that $\forall i$, $|u_i - v_i| < 1/2n^{1.5}$.

- Sort the coordinates of $\bar{u}$; wlog $u_1 \geq \cdots \geq u_n$.
- $\sum u_i = 0$ & $\sum u_i^2 = 1 \Rightarrow$
  either $u_1 \geq 1/\sqrt{n}$ or $u_n \leq -1/\sqrt{n}$.
  $\Rightarrow u_1 - u_n \geq 1/\sqrt{n}$
  $\Rightarrow \exists i_0, \quad u_{i_0} - u_{i_0+1} > 1/n^{1.5}$
  $\Rightarrow \forall i \in [i_0], j \in [i_0+1 \dots n], \ u_i - u_j > 1/n^{1.5}$.

- Since G is connected we can pick such an edge, say (i,j).

$$\Rightarrow A_{i,j} \cdot (u_i - v_j)^2 \geq \frac{1}{d} \cdot (|u_i - u_j| - |u_j - v_j|)^2$$

$$> \frac{1}{d}\left(\frac{1}{n^{1.5}} - \frac{1}{2n^{1.5}}\right)^2 = \frac{1}{4dn^3} .$$

$\Rightarrow 1 - \|Au\|^2 \geq 1/4dn^3$

$\Rightarrow 1 - \lambda(A) \geq 1/8dn^3$. □

<u>Lemma 3</u>: Let $\ell := 10dn^3 \lg n$. If s, t are connected in G then $\Pr[\text{random walk reaches } t \text{ at the } \ell\text{-th step}] > 1/2n$.

Pf: • Let $\bar{p}$ be the probability vector at the ℓ-th step.

• Lemma 2 & 1 $\Rightarrow \|A^{\ell}\cdot \bar{e}^{s} - \bar{1}\| \le \left(1-\frac{1}{8dn^3}\right)^{\ell}$

$$\le \left(1-\frac{1}{8dn^3}\right)^{10dn^3\lg n} < \frac{1}{2n^{1.5}}.$$

• By Cauchy-Schwarz inequality:

$$\|A^{\ell}\bar{e}^{s} - \bar{1}\|_1 \le \|A^{\ell}\bar{e}^{s} - \bar{1}\|_2 \cdot \sqrt{n} < 1/2n.$$

$$\Rightarrow |(A^{\ell}\cdot\bar{e}^{s} - \bar{1})_t| < 1/2n$$

$$\Rightarrow \Pr[\text{reaching } t \text{ at } \ell] > \frac{1}{n} - \frac{1}{2n} = \frac{1}{2n}. \quad \square$$

- Thus, by continuing this walk for a longer amount we can bring the probability above 3/4.

eg. $\left(1-\frac{1}{2n}\right)^{4n} \le \frac{1}{e^2} < \frac{1}{4}$

$\Rightarrow \ell = 40dn^4\log n$ suffices.

- This random walk is in logspace, as we only need to store the current vertex label (of bit-size $O(\lg n)$), proving the theorem. $\square$

- Can it be derandomized?

  This was an intriguing question for three decades, and several tools were developed.

- The basic idea is to convert the input G to G' – a graph with <u>constant</u> spectral gap, so that $\ell = O(\lg n)$ suffices to reach any vertex from s.

  Now one can exhaustively look for all $O(\lg n)$-length paths from s, in $\mathbb{L}$.

- This relation between spectral gap & connectivity motivates the following two definitions of <u>expanders</u>.

<u>Definition</u> (Algebraic): • We call a graph G an <u>(n, d, λ)-expander</u> if G is n-vertex, d-regular & $\lambda(G) \le \lambda$.

- A (d,λ)-expander family $\{G_n\}_n$ is s.t. $\forall n$, G_n is an (n,d,λ)-expander.

 - Alon-Boppana (1986) showed that $\lambda(G)$ is at least $2\sqrt{d-1}/d$.

assuming a sufficiently large diameter

 The graphs meeting this bound are called Ramanujan graphs.

 Their explicit constructions are due to Lubotzky-Phillips-Sarnak (1988).

Definition (Combinatorial): We call G an (n,d,ρ)-edge expander if G is an n-vertex d-regular graph s.t. $\forall S \subseteq V(G)$ with $|S| \leq n/2$, $|E(S,\bar{S})| \geqslant \rho \cdot d \cdot |S|$.

edges going out of S

 - Note: In the algebraic definition we desire λ to be small ($\approx 2/\sqrt{d}$), while in the combinatorial definition we want ρ to be large ($\approx 1/2$).