Expanders

- We now start the first topic in our list of pseudorandom constructions.

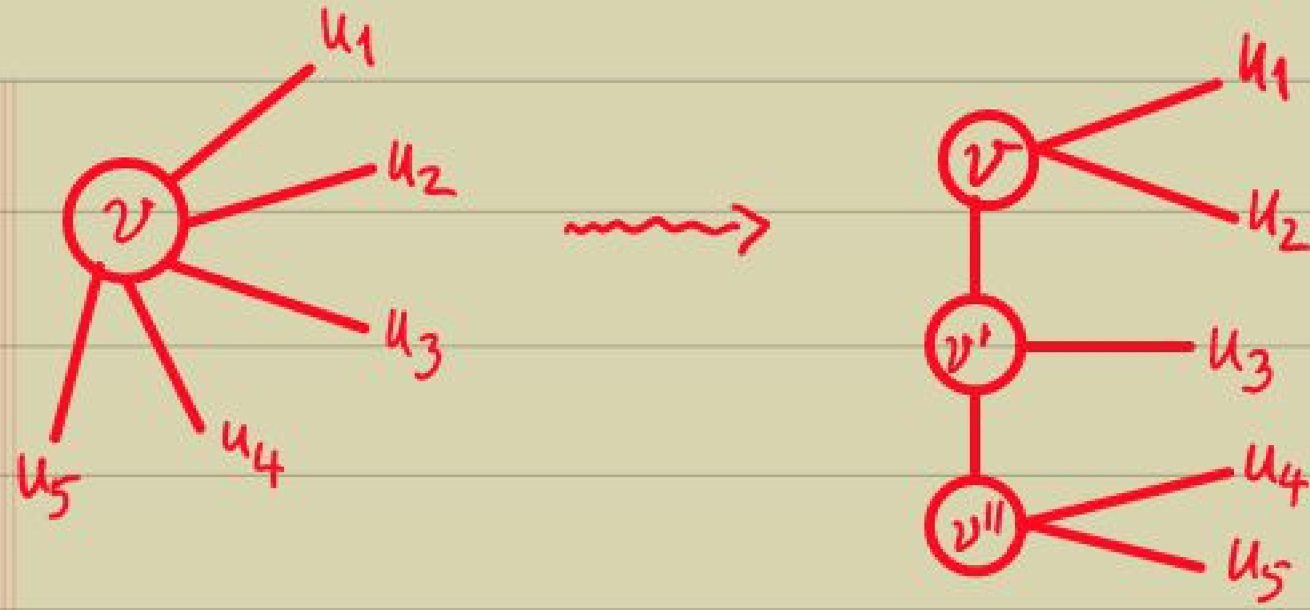
- Motivation: Solving the problem of undirected connectivity in logspace ($\mathbb{L}$ or RL).

Defn: Upath $:= \{(G,s,t) \mid \exists$ path $s \rightsquigarrow t$ in the undirected graph $G\}$.

Theorem (Aleliunas, Karp, Lipton, Lovász, Rackoff 1979):
Upath $\in RL$.

Pf:

- Suppose G is the given undirected graph with n vertices.
- Wlog we can assume G to be d-regular.
  Eg. the following gadget achieves 3-regularity (in logspace):



- Now the algorithm is to do a <u>random walk</u>, starting from v, of length $300n^4 \lg n$.

Helps in remembering the history of the walk

- Assume that each vertex in G has a <u>self-loop</u>.

- Let A be the <u>normalized</u> adjacency matrix of G. I.e. $A_{ij} := \#edges(i,j) / d$.

▷ A is symmetric with entries in $[0,1]$.

▷ The row-sum, resp. col-sum, is $\underline{1}$.

(A is <u>symmetric stochastic</u>.)

- At any stage of the walk, $\bar{p} = (p_1, \ldots, p_n)^T$ collects the probability p_i of being at the vertex i.

▷ In one step of the random walk the probability changes as $\bar{p}$ to $\bar{q} = A\cdot\bar{p}$.

Pf:

- $q_i := \Pr[\text{walk is at } i]$
$$= \sum_{j\in[n]} \Pr[\text{walk was at } j] \cdot \Pr[\text{walk is at } i \mid \text{was at } j]$$
$$= \sum_j p_j \cdot A_{ji} = (A\cdot\bar{p})_i . \qquad \square$$

- Let $\bar{e}^s$ be the elementary vector that is 1 at the s-th coordinate.

▷ After l steps of the random walk, the probability vector is $A^l\cdot\bar{e}^s$.

- Now we study the action of A, by using its eigenvalues as the main tool.

Exercise: A has real eigenvalues $\lambda_1, \dots, \lambda_n$ with $|\lambda_n| \le |\lambda_{n-1}| \le \dots \le |\lambda_1| = 1$.

– Denote the uniform probability vector $(1/n, \dots, 1/n)^T$ by $\bar{1}$.

Since $A \cdot \bar{1} = \bar{1}$ we have $\bar{1}$ as an eigenvector. Consider $\bar{1}^{\perp} := \{v \in \mathbb{R}^n \mid \langle v, \bar{1} \rangle = 0\}$.
(vectors orthogonal to $\bar{1}$)

▷ $\lambda(A) := \max\{\|Av\| \mid v \in \bar{1}^{\perp} \text{ \& } \|v\| = 1\}$ is the second largest eigenvalue of A.

Pf:

- Since A is symmetric we can find an orthonormal basis $\{b_1, \dots, b_n\}$ of $\mathbb{R}^n$ s.t. b_i is an eigenvector of λ_i and $b_1 = \bar{1}$.

$\Rightarrow \quad \bar{1}^{\perp} = sp_{\mathbb{R}}\{b_2, \dots, b_n\}$

$\Rightarrow$ for a $v = \sum_{i>1} \alpha_i b_i \in \bar{1}^{\perp}$,

$Av = \sum_{i>1} \alpha_i \lambda_i b_i$

$$\Rightarrow \frac{\|Av\|^2}{\|v\|^2} = \frac{\|\sum \alpha_i \lambda_i b_i\|^2}{\|\sum_{i>1} \alpha_i b_i\|^2} = \frac{\sum \alpha_i^2 \lambda_i^2}{\sum_{i>1} \alpha_i^2} \leq \lambda_2^2 .$$

$\Rightarrow$ max $\|Av\|$ over unit vectors in $\bar{1}^{\perp}$ is exactly λ_2. $\square$

▷ $\lambda(A^\ell) \leq \lambda(A)^\ell$.

Pf:

- By the definition of $\lambda(\cdot)$,

$$\|Av\| \leq \lambda(A)\cdot\|v\| \quad , \text{ for } v \in \bar{1}^\perp.$$

- Also, $\langle Av, \bar{1}\rangle = \langle v, A\cdot\bar{1}\rangle = \langle v, \bar{1}\rangle = 0$, for $v \in \bar{1}^\perp$.

$\Rightarrow$ A maps $\bar{1}^\perp$ to itself, shrinking each vector by a factor of $(\leq)$ $\lambda(A)$.

$\Rightarrow \|A^\ell v\| \leq \lambda(A)^\ell\cdot\|v\|$, for $v \in \bar{1}^\perp$

$\Rightarrow \lambda(A^\ell) \leq \lambda(A)^\ell$. $\square$

Exercise: $\lambda(A^\ell) = \lambda(A)^\ell$.

Lemma 1: $\forall$ probability vector $\bar{p}$, $\|A^\ell\cdot\bar{p} - \bar{1}\| < \lambda(A)^\ell$.

Pf:

- We use the eigenvector decomposition.
- Let $\bar{p} = \alpha\cdot\bar{1} + \bar{p}'$, where $\bar{p}' \in \bar{1}^\perp$, $\alpha \in \mathbb{R}$.

$\Rightarrow 1/n = \langle \bar{p}, \bar{1}\rangle = \alpha\cdot\langle\bar{1},\bar{1}\rangle + \langle\bar{p}',\bar{1}\rangle \Rightarrow \alpha = 1$.

$$\Rightarrow \quad \bar{p} = \bar{1} + \bar{p}'.$$

$$\Rightarrow \quad A^{\ell}\cdot\bar{p} = A^{\ell}\cdot\bar{1} + A^{\ell}\cdot\bar{p}' = \bar{1} + A^{\ell}\cdot\bar{p}'$$

$$\Rightarrow \quad \|A^{\ell}\cdot\bar{p} - \bar{1}\| = \|A^{\ell}\cdot\bar{p}'\| \leq \lambda(A)^{\ell}\cdot\|\bar{p}'\|$$

• Also, $\|\bar{p}\|^2 = \|\bar{1}\|^2 + \|\bar{p}'\|^2$

$$\Rightarrow \|\bar{p}'\| < \|\bar{p}\| \leq \left(\sum_{i=1}^{n} p_i\right)^2 = 1.$$

$$\Rightarrow \|A^{\ell}\cdot\bar{p} - \bar{1}\| < \lambda(A)^{\ell}. \qquad \square$$

▷ Thus, the <u>further</u> $\lambda(A)$ is from 1, the <u>faster</u> is the convergence of $A^{\ell}\cdot\bar{p}$ to $\bar{1}$!

– $1-\lambda(A)$, or $1-\lambda(G)$, is called the <u>spectral gap</u> of the graph G.
We wish it large.

<u>Lemma 2:</u> $\forall$ d-regular, connected G (with self-loops),
$$1-\lambda(G) \geq 1/8dn^3.$$