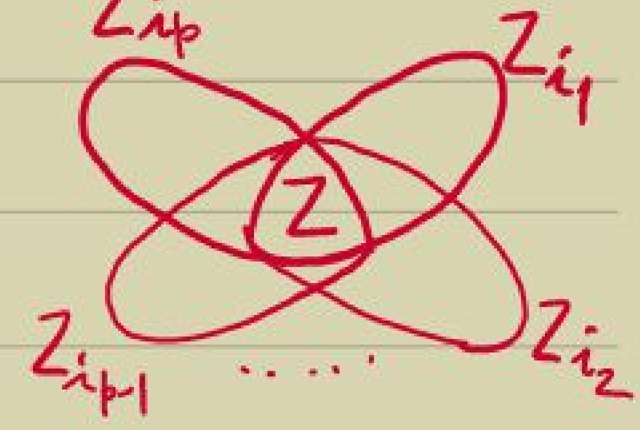
(We call such a function : an (m, ℓ)-function.)

• We construct $\tilde{f}_i$ is by induction.
For $f_i = f_{i'} \vee f_{i''}$ we construct $\tilde{f}_i := \tilde{f}_{i'} \sqcup \tilde{f}_{i''}$ as follows:
(respectively, $f_{i'} \wedge f_i'' \mapsto f_i' \sqcap f_i''$.)

• Operation $f \sqcup g$:
Let f, g be two (m,ℓ)-functions:
$f = \bigvee_{i\in[m]} C_{S_i}$ & $g = \bigvee_{j\in[m]} C_{T_j}$.

• Consider the function $h := \bigvee_{i\in[2m]} C_{Z_i}$, where $Z_i := S_i$ & $Z_{m+j} := T_j$, $\forall i, j \in [m]$.
Clearly, h is not an (m,ℓ)-function.

• We approximate h by an (m,ℓ)-function as follows:
i) As long as $\exists > m$ distinct sets, find p subsets $Z_{i_1}, \dots, Z_{i_p}$ that form a sunflower (i.e. $\exists Z \subseteq [n], \forall j < j' \in [p]$, $Z_{i_j} \cap Z_{i_{j'}} = Z$).



ii) Replace the functions $C_{Z_{i_1}}, \dots, C_{Z_{i_p}}$ in h with C_Z.

iii) Repeat this till we get an (m,ℓ)-function h'. Define $f \sqcup g := h'$.

▷ We do not get stuck because a sunflower <u>exists</u> in each of the above steps.

"<u>Sunflower" Lemma</u> (Erdös & Rado, 1960): Let $\mathcal{Z}$ be a collection of distinct sets of size $\leq \ell$. If $|\mathcal{Z}| > (p-1)^{\ell} \cdot \ell!$ then $\exists\, Z_1, \dots, Z_p \in \mathcal{Z}$ (each of size $\leq \ell$) & a set Z s.t. $\forall i < j \in [p]$, $Z_i \cap Z_j = Z$.

• How well does $f \sqcup g$ <u>approximate $f \vee g$</u> ?

▷ $\Pr_{G \in \mathcal{Y}} [(f \sqcup g)(G) < f(G) \vee g(G)] = 0.$

Pf: For any $Z \subseteq Z_i$, $C_{Z_i}(G) = 1 \Rightarrow C_Z(G) = 1$

$\Rightarrow$ If $f(G) \vee g(G) = 1$ then $(f \sqcup g)(G) = 1$.

$\square$

$\triangleright$ $\Pr_{G \in \mathcal{N}} [(f \sqcup g)(G) > f(G) \vee g(G)] < 1/10p$.

Pf: • During an application of the Sunflower's lemma, we may make an $\mathcal{N}$ instance true, by replacing $C_{Z_1}, \dots, C_{Z_p}$ by C_Z s.t. $C_Z(G) = 1$ while $\forall i \in [p]$, $C_{Z_i}(G) = 0$.

• Recall that $G \in \mathcal{N}$ is generated by choosing a random $c : [n] \to [k-1]$ & adding an edge (u,v) iff $c(u) \neq c(v)$.

$\Rightarrow$ c is 1-1 on Z but <u>not</u> on Z_i's.

• $\Pr_c [c \text{ is 1-1 on } Z_i \mid c \text{ is 1-1 on } Z]$

$$\geq \left(1 - \frac{|Z|}{k-1}\right)\left(1 - \frac{|Z|+1}{k-1}\right) \cdots \left(1 - \frac{\ell}{k-1}\right)$$

$$> 1 - \frac{|Z| + \dots + \ell}{k-1} > 1 - \frac{\ell^2}{k-1} > 1/2$$

- As $Z_1 \setminus Z, \dots, Z_p \setminus Z$ are mutually disjoint we also get:

$$\Pr_c[\forall i \in [p], \ c \text{ is not 1-1 on } Z_i \mid c \text{ is 1-1 on } Z]$$
$$< \left(\tfrac{1}{2}\right)^p = n^{-10\sqrt{k}} < 1/10ms. \quad \left(\begin{matrix} m, s \\ < n^{\sqrt{k}/20} \end{matrix}\right)$$

- Sunflower lemma might be applied $< m$ times

$$\Rightarrow \Pr_{G \in N}[(f \sqcup g)(G) \text{ is wrong}] < m \cdot \frac{1}{10ms} = \frac{1}{10s}.$$

$\square$

<u>Operation $f \sqcap g$</u>:

- Let h be the function $\bigvee_{i,j \in [m]} C_{S_i \cup T_j}$.

- We approximate it by an (m, ℓ)-function as:

i) Drop those C_Z from h s.t. $|Z| > \ell$.

ii) Reduce the number of clique indicators by repeated applications of the Sunflower lemma.

iii) Remaining function h' is $(f \sqcap g)$.

- How **well** does $f \sqcap g$ approximate $f \wedge g$?

▷ $\Pr_{G \in \mathcal{Y}} [(f \sqcap g)(G) < f(G) \wedge g(G)] < 1/10s.$

Pf: • $f = \bigvee_{i \in [m]} C_{S_i}$ & $g = \bigvee_{j \in [m]} C_{T_j}$.

* $\Rightarrow f \wedge g = \bigvee_{i,j \in [m]} C_{S_i \cup T_j} = h.$

• A $G \in \mathcal{Y}$ corresponds to a $K \in \binom{[n]}{k}$.

• The only way $(f \sqcap g)(G) = 0$, while $f(G) \wedge g(G) = 1$ for a $G \in \mathcal{Y}$, is if we drop a C_Z, $Z \subseteq K$ & $|Z| > \ell$. $(\ell = \sqrt{k}/10)$

• By Lemma 1, $\Pr_{K \in \binom{[n]}{k}} [Z \subseteq K] < n^{-0.7\ell} < \frac{1}{10 s m^2}$.

• We could drop at most m^2 Z's from h.

$\Rightarrow \Pr_{G \in \mathcal{Y}} [(f \sqcap g)(G) \text{ is wrong}] < 1/10s.$ $\square$

* Prove that $C_{S_i} \wedge C_{T_j} = C_{S_i \cup T_j}$ on $G \in \mathcal{Y}$.

▷ $\Pr_{G \in \mathcal{N}} [(f \sqcap g)(G) > f(G) \wedge g(G)] < 1/10s$.

Pf: • A $G \in \mathcal{N}$ corresponds to a $c : [n] \to [k-1]$.

• The only way $(f \sqcap g)(G) = 1$, while $f(G) \wedge g(G) = 0$ for $G \in \mathcal{N}$, is when we replace $C_{Z_1}, \ldots, C_{Z_p}$ by C_Z s.t. c is 1-1 on Z but c is <u>not</u> 1-1 on Z_i, $\forall i \in [p]$.

$\Rightarrow$ an analysis like that of $f \sqcup g$ gives us error probability $< 1/10s$. $\square$

▷ As we compute $\sqcup$ & $\sqcap$ at most s times in C, we get:

$$\Pr_{G \in \mathcal{Y}} [\tilde{f}_s(G) < C(G)] < 1/10 \quad \&$$

$$\Pr_{G \in \mathcal{N}} [\quad " \quad > \quad " \quad] < 1/10.$$

$\Rightarrow$ This finishes Lemma 2. $\square$

– Finally, we prove the Sunflower lemma:

Sunflower lemma

Pf: • We induct on ℓ (set size-bound).

• For $\ell=1$, $\mathcal{Z}$ has only singletons.

$\Rightarrow$ any distinct $Z_1, \ldots, Z_p \in \mathcal{Z}$ is a sunflower.

- Let $\ell > 1$.

Let $\mathcal{M}$ be a maximal collection of mutually disjoint sets in $\mathcal{Z}$.

If $|\mathcal{M}| \geq p$ then we are done.

• $|\mathcal{M}| < p \Rightarrow |\cup \mathcal{M}| \leq (p-1)\cdot \ell$.

• Also, $\mathcal{M}$'s maximality $\Rightarrow \forall Z \in \mathcal{Z}$, $Z \cap (\cup \mathcal{M}) \neq \phi$.

$\Rightarrow \exists x \in \cup \mathcal{M}$ appearing in $s \geq |\mathcal{Z}| / \ell(p-1)$ many sets in $\mathcal{Z}$, say $Z_1, \ldots, Z_s \in \mathcal{Z}$.

• Since $s > (\ell-1)! \cdot (p-1)^{\ell-1}$, by induction $\exists$ a sunflower in $\{Z_1 \setminus \{x\}, \ldots, Z_s \setminus \{x\}\}$

$\Rightarrow \exists$ a sunflower in $\mathcal{Z}$. $\square$

- Sunflower conjecture: The $(\ell!)$ can be improved to c^{ℓ} for some constant c.