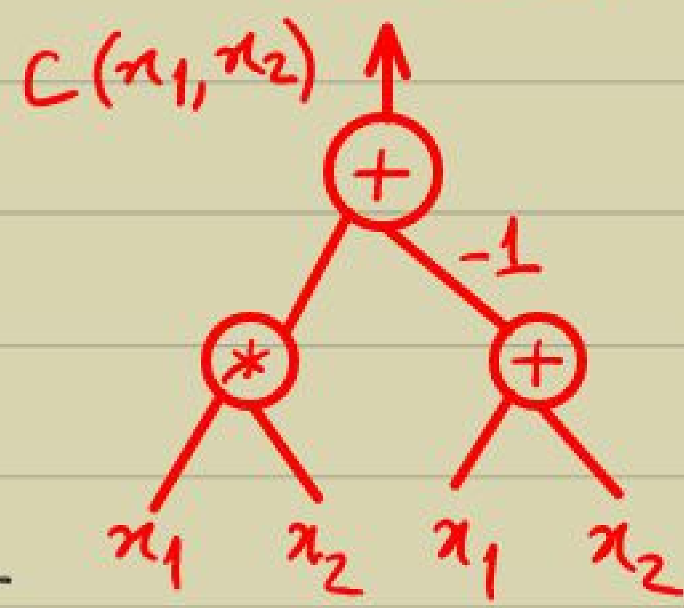
- PIT: Given an arithmetic circuit $C(x_1, .., x_n)$ over $\mathbb{F}$, test if $C = 0$?

- Arithmetic circuit: A rooted tree with inputs as leaves, output as root, $+, *$ as internal nodes, & constants on edges.



- Size of a circuit includes the #edges, #gates, representation-size of the constants.

- Circuit $C(\bar{x})$ can capture very large polynomials, in small size!

- Thus, PIT is nontrivial.

We want a poly-time algorithm for PIT over fields $\mathbb{Q}, \mathbb{F}_q$.

__Theorem:__ PIT $\in$ BPP.

__Pf:__

- Let $C(\bar{x})$ be the given circuit, over $\mathbb{F}$, of size s.
- Note that the total-deg of C is $< s^s$.
  (Each multiplication layer increases the degree by a multiple of at most s.)
- We could assume $|\mathbb{F}| > 2\cdot s^s$, otherwise we use an appropriate field extension as $\mathbb{F}$.
- The algorithm is simply a random evaluation
  (0) Pick a subset $S \subseteq \mathbb{F}$, $|S| = 2\cdot s^s$.
  (1) Pick a __random__ $(a_1, \ldots, a_n) \in S^n$.
  (2) If $C(a_1, \ldots, a_n) = 0$ then OUTPUT __zero__, else OUTPUT __nonzero.__

- Correctness:
  If $C(\bar{x}) = 0$ then $\mathrm{Prob}_{\bar{a}}[\text{correct output}] = 1$.
  Else $\mathrm{Prob}_{\bar{a}}[\text{correct output}] > 1 - \frac{s^s}{2\cdot s^s} = \frac{1}{2}$
is given by the following lemma. $\square$

Lemma (DeMillo & Lipton '78, Zippel '79, Schwartz '80): Let $P \in \mathbb{F}[\bar{x}]$ be a polynomial of degree $d \geq 0$. Let $S \subseteq \mathbb{F}$ be a finite subset. Then,

$$\Pr_{\bar{a} \in S^n} [P(\bar{a}) = 0] \leq d/|S| .$$

Proof: • For $n=1$, it follows from the fact that $P(x_1)$ can have at most d roots in $\mathbb{F}$.

Base case

• Assume it to be true for $(n-1)$ variables.

Induction hypothesis

• Write $P = \sum_{i=0}^{d} x_n^i \cdot P_i(x_1, \ldots, x_{n-1})$.

• As $P \neq 0$, let i_0 be the largest i s.t. $P_i \neq 0$.

$a_1, \ldots, a_{n-1}$

$$\Rightarrow \Pr_{\bar{a}} [P(\bar{a}) = 0] = \Pr_{\bar{a}} [P_i(\bar{a}) = 0] \cdot \Pr_{\bar{a}} [P(\bar{a}) = 0 \mid P_i(\bar{a}) = 0]$$

$$+ \Pr_{\bar{a}} [P_i(\bar{a}) \neq 0] \cdot \Pr_{\bar{a}} [P(\bar{a}) = 0 \mid P_i(\bar{a}) \neq 0]$$

$$\leq \Pr [P_i(\bar{a}) = 0] + \Pr [P(\bar{a}) = 0 \mid P_i(\bar{a}) \neq 0]$$

induction

univariate

$$\leq \frac{d - i_0}{|S|} + \frac{i_0}{|S|} = \frac{d}{|S|} . \qquad \square$$

- For the small field case show that:

<u>Exercise</u>: A field $G \supset \mathbb{F}$ of size $> 2 \cdot s^{D}$ can be constructed in randomized poly(s)-time.

The Circuit Model

- An arithmetic circuit (over $\mathbb{F}$) has $+, *$ gates and field elements.

- A boolean circuit has AND, OR, NOT gates and $\{0,1\}$. (0=false, 1=true)

- An arithmetic circuit outputs a polynomial while a boolean circuit outputs a boolean formula.

- We can use these as a model of computation instead of TMs.

Defn: • A problem $L \subseteq \{0,1\}^*$ is said to be solved by a boolean circuit family $\{C_n(x_1,\dots,x_n) \mid n \geq 1\}$ if $\forall n$, $\forall x \in \{0,1\}^n$, $C_n(x) = 1$ iff $x \in L$.

• The computational resources now are: size(C_n), depth(C_n) & fanin / fanout.

Proposition: (1) Any TM can be turned into a circuit family (vice versa?)

(2) Boolean circuits are inspired from "electronics" & capture parallel computation:

size(C) corresponds to the space requirement of a parallel algorithm.

depth(C) corresponds to the time taken by the parallel algorithm.

(3) Two n-bit integers can be added by a size-poly(n), constant-depth boolean circuit.