- Observe that $\dim L_i/L_{i-1} \leq d(P) = 1$.

$\Rightarrow \exists$ a k-basis $\{f_1, \ldots, f_r\}$, $r \leq a$, of L_a st. $\forall i$, $v_p(f_i) < v_p(f_{i+1})$.

[$\because$ We can pick an $f_i \in L_{j_i} \setminus L_{j_i - 1}$ whenever possible, and order the f_i's suitably.]

- Now write any element in $L_b^{p,\mu} \cdot L_a^{\varphi}$ as:

$$G = \sum_{1 \leq i \leq r} (s_i F_{a+b}^{\mu}) \cdot (f_i \varphi), \text{ where } s_i \in L_b.$$

- When is $G = 0$?
- Suppose s_h is the first <u>nonzero</u> s_i.
- Taking v_p on:

$$-(s_h F_{a+b}^{\mu}) \cdot (f_h \varphi) = \sum_{h+1 \leq i \leq r} (s_i F_{a+b}^{\mu}) \cdot (f_i \varphi)$$

$$\Rightarrow p^{\mu} \cdot v_p(s_h) + q \cdot v_p(f_h) \geq \min_{i > h} \{ p^{\mu} \cdot v_p(s_i) + q \cdot v_p(f_i) \}.$$

$$\geq p^{\mu} \cdot (-b) + q \cdot v_p(f_{h+1})$$

$$\Rightarrow p^{\mu} \cdot v_p(s_h) \geq -bp^{\mu} + q \cdot (v_p(f_{h+1}) - v_p(f_h))$$

$$\geq -bp^{\mu} + q > 0.$$

- Thus, s_h vanishes at P and has no pole.

$\Rightarrow s_h = 0$, a contradiction! $\square$

- Applying Claim 1, we get a well-defined map τ:

$$\begin{tikzcd} L_b^{p,\mu} \cdot L_a^{\phi} \arrow[r, "\tau"] & L_b^{p,\mu} \cdot L_a \arrow[r, hook] & L_{bp^\mu + a} \\ L_b^{p,\mu} \otimes_k L_a^{\phi} \arrow[u, "\wr", "\text{(Claim 1)}"'] & L_b^{p,\mu} \otimes_k L_a \arrow[l] \arrow[u, "\text{multiplication}"'] & \end{tikzcd}$$

- Let us assume the setting:

$$bp^\mu < q \ \& \ \ell_b \ell_a > \ell_{bp^\mu + a}.$$

- By the second inequality, we deduce that $\ker(\tau)$ has a nonzero element, say G:

$$G = \sum_{i \in [r]} (s_i F_{abs}^{\mu}) \cdot (f_i \phi) \quad \text{s.t.}$$

$$\tau(G) = \sum_i (s_i F_{abs}^{\mu}) \cdot f_i = 0.$$

- Firstly, since $p^\mu < q$, G is a p^μ-th power itself.