- <u>Defn:</u> We denote $Z(q^{-s})$ by $\underline{\zeta(s, C)}$.

- Note that $\sum_{D \geq 0} t^{d(D)} = \prod_{P \in C} (1 - t^{d(P)})^{-1}$.

$\Rightarrow Z(q^{-s}) = \prod_{P \in C} (1 - q^{-d(P) \cdot s})^{-1}$.

- We define the <u>norm of D</u> as $\underline{N(D)} := q^{d(D)}$.

- <u>Proposition:</u> $\zeta(s, C) = \sum_{D \geq 0} N(D)^{-s} = \prod_{P \in C} (1 - N(P)^{-s})^{-1}$,

and they are all convergent when $Re(s) > 1$.

- These are analogous to the ordinary <u>Riemann zeta fn.</u>, <u>Dirichlet series</u> & the <u>ruler product</u> as:

$\zeta(s) := \sum_{n \in \mathbb{N}} \frac{1}{n^s} = \prod_{\text{prime } p} \left(1 - \frac{1}{p^s}\right)^{-1}$ when

$Re(s) > 1$, $s \in \mathbb{C}$.

The Functional equation

- We now show that $Z(t)$ is a rational fn.! Further, we show a symmetry.

(Riemann-Roch makes all this possible!)

Theorem: (i) $Z(t) \in \mathbb{Q}(t)$.

(ii) $Z(t) = (qt^2)^{g-1} \cdot Z(1/qt)$.

Proof: • $(q-1) \cdot Z(t) = \sum_{d, \mathcal{D}} (q^{\ell(\mathcal{D})} - 1) \cdot t^d$

$$= \sum_{d, \mathcal{D}} q^{\ell(\mathcal{D})} \cdot t^d - \sum_{d, \mathcal{D}} t^d$$

• We will break this into:

$$\underline{F(t)} := \sum_{\substack{d \geq 2g-2+\delta, \\ \mathcal{D}}} q^{\ell(\mathcal{D})} \cdot t^d - \sum_{d, \mathcal{D}} t^d \text{, and}$$

$$\underline{G(t)} := \sum_{\substack{0 \leq d \leq 2g-2, \\ \mathcal{D}}} q^{\ell(\mathcal{D})} \cdot t^d .$$

• Riemann-Roch says that: canonical divisor of C

$$\ell(\mathcal{D}) - d(\mathcal{D}) = 1 - g + \ell(W - \mathcal{D}).$$

$\Rightarrow \ell(0) = 1 - g + \ell(W) \Rightarrow \underline{\ell(W) = g}$.

• Also, $\ell(W) - d(W) = 1 - g + \ell(0) \Rightarrow \underline{d(W) = 2g-2}$.

• $\Rightarrow (2g-2) \in \delta\mathbb{N}$.

• When $d(\mathcal{D}) > 2g-2$, then $d(W-\mathcal{D}) = d(W) - d(\mathcal{D})$
$= (2g-2) - d(\mathcal{D}) < 0$. So, $\ell(W-\mathcal{D}) = 0$,
$\Rightarrow$ For such $\mathcal{D}$'s : $\ell(\mathcal{D}) = d(\mathcal{D}) + 1 - g$.

• This greatly helps in simplifying $F(t)$:

$$F(t) = \sum_{\delta\mathbb{N} \ni d \geq 2g-2+\delta} h(C)\cdot q^{d+1-g}\cdot t^d - \frac{h(C)}{1-t^\delta}$$

$$= h(C)\cdot q^{1-g}\cdot \sum_{2g-2+\delta \leq d \in \delta\mathbb{N}} (qt)^d - \frac{h(C)}{1-t^\delta}$$

$$= h(C)\cdot q^{1-g}\cdot \frac{(qt)^{2g-2+\delta}}{1-(qt)^\delta} - \frac{h(C)}{1-t^\delta}$$

• Since, obviously, $G(t)$ is an <u>integral</u> polynomial, we get that $Z(t)$ is in $\mathbb{Q}(t)$.

• For the symmetry of $Z(t)$, observe :

$$F((qt)^{-1}) = h(C)\cdot q^{1-g}\cdot \frac{(t^{-1})^{2g-2+\delta}}{1-t^{-\delta}} - \frac{h(C)}{1-(qt)^{-\delta}}$$

$$= (qt^2)^{1-g}\left\{\frac{h(C)}{t^\delta - 1} - \frac{h(C)\cdot(qt)^\delta\cdot(qt^2)^{g-1}}{(qt)^\delta - 1}\right\}$$

$$= (qt^2)^{1-g}\cdot F(t).$$

• On the other hand,

$$G((qt)^{-1}) = \sum_{\substack{\mathbb{N} \ni d \leq 2g-2, \\ \mathcal{D}}} q^{\ell(\mathcal{D})} \cdot (qt)^{-d}$$

$$= \sum_{d, \mathcal{D}} q^{d+1-g+\ell(W-\mathcal{D})} \cdot (qt)^{-d}$$

$$= (qt^2)^{1-g} \cdot \sum_{d,\mathcal{D}} q^{\ell(W-\mathcal{D})} \cdot t^{2g-2-d}.$$

$$= (qt^2)^{1-g} \cdot \sum_{d,\mathcal{D}} q^{\ell(W-\mathcal{D})} \cdot t^{d(W-\mathcal{D})}$$

• Note that as d runs over $[0, 2g-2]$, $(2g-2-d)$ runs over $(0, 2g-2)$ as well!

$\Rightarrow G(1/qt) = (qt^2)^{1-g} \cdot G(t)$.

• Combining the "symmetry" of F & G, we get the same for $Z(t)$. $\square$

- <u>Corollary 1</u>: $\zeta(1-s, C) = N(W)^{s-\frac{1}{2}} \cdot \zeta(s, C)$.

This expresses the mysterious <u>symmetry about the $Re(s)=\frac{1}{2}$ axis</u>!

- Corollary 2: The poles of $Z(t)$ are given by $(1-t^\delta)(1-q^\delta t^\delta)=0$. (They are simple.)

- We would like to study $Z(t)$ as the base field k varies. So, we now use $Z(t,C)$.

- Theorem (Base change): Let C_n be the curve obtained from C by extending $k=\mathbb{F}_q$ to $k'=\mathbb{F}_{q^n}$. Then,

$$Z(t^n, C_n) = \prod_{\eta^n=1} Z(\eta t, C) .$$

Proof: • Idea: Work with Euler product. Thus, it suffices to compare a point $P\in C$ with its conjugates in C_n.

• For $P\in C$, let $m_P \triangleleft R_P$ be the dvr data.

• We have $k_P = R_P/m_P$ as a field extn. of k of degree $d(P)$.

• Pick $x\in R_P$ whose image $\bar{x}\in k_P$ generates k_P over k. I.e. $k_P \cong k[T]/(f)$, where $f(T)$ is the $\mathrm{minpoly}_k(\bar{x})$.